Functions
Outline for Today

- **What is a Function?**
  - It’s more nuanced than you might expect.
- **Domains and Codomains**
  - Where functions start, and where functions end.
- **Defining a Function**
  - Expressing transformations compactly.
- **Special Classes of Functions**
  - Useful types of functions you’ll encounter IRL.
- **Proofs on First-Order Definitions**
  - A key skill.
What is a function?
Functions, High-School Edition
\[ f(x) = x^4 - 5x^2 + 4 \]
\[ f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21} \]
Functions, High-School Edition

- In high school, functions are usually given as objects of the form

\[ f(x) = \frac{x^3 + 3x^2 + 15x + 7}{1 - x^{137}} \]

- What does a function do?
  - It takes in as input a real number.
  - It outputs a real number.
  - ... except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.
Functions, CS Edition
```c
int flipUntil(int n) {
    int numHeads = 0;
    int numTries = 0;

    while (numHeads < n) {
        if (randomBoolean()) {
            numHeads++;
        }
        numTries++;
    }

    return numTries;
}
```
Functions, CS Edition

- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.
What's Common?

• Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  • They take in inputs.
  • They produce outputs.
• In math, we like to keep things easy, so that's pretty much how we're going to define a function.
High-Level Intuition:

A function is an object $f$ that takes in exactly one input $x$ and produces exactly one output $f(x)$.

(This is not definition. It’s just to help you build and intuition.)
High School versus CS Functions

• In high school, functions usually were given by a rule:
  \[ f(x) = 4x + 15 \]

• In CS, functions are usually given by code:
  ```c
  int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
      result *= i;
    }
    return result;
  }
  ```

• What sorts of functions are we going to allow from a mathematical perspective?
... but also ...
\[ f(x) = x^2 + 3x - 15 \]
In mathematics, functions are *deterministic*. That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it’s not a valid function under our definition:

```cpp
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```
One Challenge
\[ f(x) = x^2 + 2x + 5 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 3 \cdot 2 + 5 = 20 \]
\[ f(x) = x^2 + 2x + 5 \]

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\[ f(0) = 0^2 + 0 \cdot 2 + 5 = 5 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 3 \cdot 2 + 5 = 20 \]

\[ f(0) = 0^2 + 0 \cdot 2 + 5 = 5 \]

\[ f(\text{Pikachu}) = \ldots ? \]
\[ f(\text{Pikachu}) = \text{Pikachu} \]
\[ f(137) = \ldots? \]
We need to make sure we can't apply functions to meaningless inputs.
Domains and Codomains

- Every function $f$ has two sets associated with it: its **domain** and its **codomain**.

- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.

![Diagram of domains and codomains]

- The function must be defined for every element of the domain.
- The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.
Domains and Codomains

- Every function $f$ has two sets associated with it: its **domain** and its **codomain**.
- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.

```java
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

The **domain** of this function is $\mathbb{R}$. Any real number can be provided as input.

The **codomain** of this function is $\mathbb{R}$. Everything produced is a real number, but not all real numbers can be produced.
Domains and Codomains

- If $f$ is a function whose domain is $A$ and whose codomain is $B$, we write $f : A \rightarrow B$.
- Think of this like a “function prototype” in C++. 

```
B f(A arg);
```

```
f : A \rightarrow B
```
Domains and Codomains

• If \( f \) is a function whose domain is \( A \) and whose codomain is \( B \), we write \( f : A \rightarrow B \).

• Think of this like a “function prototype” in C++.
The Official Rules for Functions

• Formally speaking, we say that $f : A \to B$ if the following two rules hold.

• First, $f$ must be obey its domain/codomain rules:

$$\forall a \in A. \exists b \in B. \; f(a) = b$$

("Every input in A maps to some output in B.")

• Second, $f$ must be deterministic:

$$\forall a_1 \in A. \forall a_2 \in A. \; (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

("Equal inputs produce equal outputs.")

• If you’re ever curious about whether something is a function, look back at these rules and check! For example:
  • Can a function have an empty domain?
  • Can a function have an empty codomain?
Defining Functions
Defining Functions

• To define a function, you need to
  • specify the domain,
  • specify the codomain, and
  • give a rule used to evaluate the function.
• All three pieces are necessary.
  • We need to domain to know what the function can be applied to.
  • We need to codomain to know what the output space is.
  • We need the rule to be able to evaluate the function.
• There are many ways to do this. Let’s go over a few examples.
Functions can be defined as a picture. Draw the domain and codomain explicitly. Then, add arrows to show the outputs.
Functions can be defined as a **rule**. Be sure to explicitly state what the domain and codomain are!

\[ f : \mathbb{Z} \to \mathbb{Z}, \text{ where} \]
\[ f(x) = x^2 + 3x - 15 \]
Some rules are given piecewise. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

\[ f : \mathbb{Z} \rightarrow \mathbb{N}, \text{ where} \]

\[ f(n) = \begin{cases} 
  n & \text{if } n \geq 0 \\
  -n & \text{if } n \leq 0 
\end{cases} \]
Some Nuances
Is this a function from $A$ to $B$?
Is this a function from $A$ to $B$?
عيد الفطر
عيد الأضحى
بيع الأو
بيع الثاني
مادى الأولى
مادى الآخرة
ذو القعدة
ذو الحجة
Special Types of Functions
Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to $A$.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. (Yes, really!)
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Theoretically unbreakable encryption (one-time pads).
  - Transmitting a large file to multiple receivers (fountain codes).
Involutions

A function $f : A \rightarrow A$ from a set back to itself is called an **involution** if the following first-order logic statement is true about $f$:

$$\forall x \in A. \ f(f(x)) = x.$$  

("Applying $f$ twice is equivalent to not applying $f$ at all.")

Involutions have lots of interesting properties. Let’s explore them and see what we can find.
Involutions

• Which of the following are involutions?
  • \( f : \mathbb{Z} \to \mathbb{Z} \) defined as \( f(x) = x \).
  • \( f : \mathbb{Z} \to \mathbb{Z} \) defined as \( f(x) = -x \).
  • \( f : \mathbb{R} \to \mathbb{R} \) defined as \( f(x) = \frac{1}{x} \).
  • \( f : \mathbb{N} \to \mathbb{N} \) defined as follows:

\[
f(n) = \begin{cases} 
  n+1 & \text{if } n \text{ is even} \\
  n-1 & \text{if } n \text{ is odd}
\end{cases}
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Involutions

Which of the following are involutions?

- $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yeaaaah!*
- $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
- $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x}$.
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Which of the following are involutions?

- \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) defined as \( f(x) = x \). *Yep!*
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A function \( f : A \to A \) is called an \textbf{involution} if the following first-order logic statement is true about \( f \):

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• \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined as \( f(x) = \frac{1}{x} \). *Not a function!*
• \( f : \mathbb{N} \rightarrow \mathbb{N} \) defined as follows:
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Involutions, Visually

A function \( f : A \rightarrow A \) is called an **involution** if the following first-order logic statement is true about \( f \):

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\forall x \in A. \ f(f(x)) = x.
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A function $f : A \to A$ is called an **involution** if the following first-order logic statement is true about $f$:

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Proofs on Involutions
Theorem: The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as

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What does it mean for $f$ to be an involution?
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\[ \forall n \in \mathbb{Z}. \ f(f(n)) = n. \]

Therefore, we’ll have the reader pick some \( n \in \mathbb{Z} \), then argue that \( f(f(n)) = n \).
**Theorem:** The function \( f : \mathbb{Z} \to \mathbb{Z} \) defined as

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**Proof:** Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

*Case 1:* $n$ is even.

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**Proof:** Pick some \( n \in \mathbb{Z} \). We need to show that \( f(f(n)) = n \). To do so, we consider two cases.

Case 1: \( n \) is even. Then \( f(n) = n+1 \), which is odd.

Case 2: \( n \) is odd.

In either case, we see that \( f(f(n)) = n \), which is what we need to show. ■
**Theorem:** The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as

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*Case 1:* $n$ is even. Then $f(n) = n+1$, which is odd. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

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*Case 2:* $n$ is odd. Then $f(n) = n - 1$, which is even. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$. 
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**Proof:** Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

*Case 1:* $n$ is even. Then $f(n) = n+1$, which is odd. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

*Case 2:* $n$ is odd. Then $f(n) = n - 1$, which is even. Then we see that $f(f(n)) = f(n-1) = (n-1) + 1 = n$.

In either case, we see that $f(f(n)) = n$, which is what we need to show. ■
Theorem: The function \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) defined as

\[
f(n) = \begin{cases} 
  n+1 & \text{if } n \text{ is even} \\
  n-1 & \text{if } n \text{ is odd}
\end{cases}
\]

is an involution.

Proof: Pick some \( n \in \mathbb{Z} \). We need to show that \( f(f(n)) = n \). To do so, we consider two cases.

Case 1: \( n \) is even. Then \( f(n) = n+1 \), which is odd. This means that \( f(f(n)) = f(n+1) = (n+1) - 1 = n \).

Case 2: \( n \) is odd. Then \( f(n) = n - 1 \), which is even. Then we see that \( f(f(n)) = f(n - 1) = (n - 1) + 1 = n \).

In either case, we see that \( f(f(n)) = n \), which is what we need to show. \( 
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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It’s written in plain English, just as usual.
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Time-Out for Announcements!
Problem Set

- Problem Set 1 solutions are up on the course website – please take a look at them as soon as possible.
- TAs are working hard on grading your assignments. We’re aiming to have those returned to you by Wednesday morning.
Back to CS103!
Another Class of Functions
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune
Pluto
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune
Injective Functions

• A function $f : A \rightarrow B$ is called injective (or one-to-one) if the following statement is true about $f$:
  
  $\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$

  (“If the inputs are different, the outputs are different.”)

• The following first-order definition is equivalent (why?) and is often useful in proofs.
  
  $\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$

  (“If the outputs are the same, the inputs are the same.”)

• A function with this property is called an injection.

• How does this compare to our second rule for functions?
Injections

• Let be the set of all CS103 students. Which of the following are injective?

1) $f: \rightarrow \mathbb{N}$ where $f(x)$ is $x$’s Stanford ID number.

2) $f: \rightarrow \mathbb{C}$, where $\mathbb{C}$ is the set of all countries and $f(x)$ is $x$’s country of birth.

3) $f: \rightarrow \mathbb{G}$, where $\mathbb{G}$ is the set of all given (first) names, where $f(x)$ is $x$’s given (first) name.

A function $f : A \rightarrow B$ is **injective** if either statement is true:

\[
\forall x_1 \in A. \forall x_2 \in A. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
\]

\[
\forall x_1 \in A. \forall x_2 \in A. (f(x_1) = f(x_2) \rightarrow x_1 = x_2)
\]
Injections

- Let be the set of all CS103 students. Which of the following are injective?

1) \( f : \mathbb{N} \rightarrow \mathbb{N} \) where \( f(x) \) is \( x \)'s Stanford ID number.

2) \( f : \mathbb{R} \rightarrow \mathbb{R} \), where \( f(x) \) is \( x \)'s country of birth.

3) \( f : \mathbb{R} \rightarrow \mathbb{R} \), where \( f(x) \) is \( x \)'s given (first) name.

A function \( f : A \rightarrow B \) is \textit{injective} if either statement is true:

\[
\forall x_1 \in A. \forall x_2 \in A. \ (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
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\[
\forall x_1 \in A. \forall x_2 \in A. \ (f(x_1) = f(x_2) \rightarrow x_1 = x_2)
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Injections

• Let $\text{be the set of all CS103 students.}$

Which of the following are injective?

1) $f: \text{ } \to \mathbb{N}$ where $f(x)$ is $x$’s Stanford ID number.

2) $f: \text{ } \to \text{ }$, where $\text{ }$ is the set of all countries and $f(x)$ is $x$’s country of birth.

3) $f: \text{ } \to \text{ }$, where $\text{ }$ is the set of all given (first) names, where $f(x)$ is $x$’s given (first) name.

A function $f: A \to B$ is injective if either statement is true:

$$\forall x_1 \in A. \forall x_2 \in A. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\forall x_1 \in A. \forall x_2 \in A. (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$$
**Injections**

- Let the set of all CS103 students be $\mathcal{S}$.

Which of the following are injective?

1) $f : \mathcal{S} \rightarrow \mathbb{N}$ where $f(x)$ is $x$’s Stanford ID number.

2) $f : \mathcal{S} \rightarrow \mathcal{C}$, where $\mathcal{C}$ is the set of all countries and $f(x)$ is $x$’s country of birth.

3) $f : \mathcal{S} \rightarrow \mathcal{G}$, where $\mathcal{G}$ is the set of all given (first) names, where $f(x)$ is $x$’s given (first) name.

A function $f : A \rightarrow B$ is **injective** if either statement is true:

$\forall x_1 \in A. \forall x_2 \in A. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$

$\forall x_1 \in A. \forall x_2 \in A. (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$

Injections

- Let be the set of all CS103 students. Which of the following are injective?

1) $f : P \to \mathbb{N}$ where $f(x)$ is $x$’s Stanford ID number.

2) $f : M \to P$, where $P$ is the set of all countries and $f(x)$ is $x$’s country of birth.

3) $f : N \to P$, where $P$ is the set of all given (first) names, where $f(x)$ is $x$’s given (first) name.

A function $f : A \to B$ is injective if either statement is true:

- $\forall x_1 \in A. \forall x_2 \in A. \ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$
- $\forall x_1 \in A. \forall x_2 \in A. \ (f(x_1) = f(x_2) \to x_1 = x_2)$
Injective Functions

**Theorem:** Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.
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**Proof:**

What does it mean for the function \( f \) to be injective?
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**Proof:**

What does it mean for the function \( f \) to be injective?

\[
\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \left( f(n_1) = f(n_2) \rightarrow n_1 = n_2 \right)
\]

\[
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\]

Therefore, we'll pick arbitrary \( n_1, n_2 \in \mathbb{N} \), assume \( f(n_1) = f(n_2) \), then prove that \( n_1 = n_2 \).
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Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$, assume $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$. 

$\therefore n_1 = n_2$. ■
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Injective Functions

**Theorem:** Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.

**Proof:** Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$. 
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Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$
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This in turn means that

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so $n_1 = n_2$, as required.
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Since \( f(n_1) = f(n_2) \), we see that

\[
2n_1 + 7 = 2n_2 + 7.
\]

This in turn means that

\[
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\]

so \( n_1 = n_2 \), as required. ■

Good exercise: Repeat this proof using the other definition of injectivity!
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**Theorem:** Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then $f$ is not injective.

Proof: We will prove that there exist integers $x_0$ and $x_1$ such that $x_0 \neq x_1$, but $f(x_0) = f(x_1)$.

Let $x_0 = -1$ and $x_1 = +1$. Then $f(x_0) = f(-1) = (-1)^4 = 1$ and $f(x_1) = f(1) = 1^4 = 1$, so $f(x_0) = f(x_1)$ even though $x_0 \neq x_1$, as required. ■
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What does it mean for \( f \) to be injective?

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\forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
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What is the negation of this statement?

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\neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
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Therefore, we need to find \( x_1, x_2 \in \mathbb{Z} \) such that \( x_1 \neq x_2 \), but \( f(x_1) = f(x_2) \). Can we do that?
Injective Functions

**Theorem:** Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then $f$ is not injective.

**Proof:** We will prove that there exist integers $x_1$ and $x_2$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Let $x_0 = -1$ and $x_1 = 1$. Then $f(x_0) = f(-1) = (-1)^4 = 1$ and $f(x_1) = f(1) = 1^4 = 1$, so $f(x_0) = f(x_1)$ even though $x_0 \neq x_1$, as required. ■
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Let $x_1 = -1$ and $x_2 = +1$. Then $f(-1) = (-1)^4 = 1$ and $f(1) = 1^4 = 1$, so $f(-1) = f(1)$ even though $x_1 \neq x_2$, as required. ■
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Let \( x_1 = -1 \) and \( x_2 = +1 \). Notice that

\[
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\[ f(x_1) = f(-1) = (-1)^4 = 1 \]

and

\[ f(x_2) = f(1) = 1^4 = 1, \]

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.
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Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■
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Another Class of Functions
Surjective Functions

• A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if this first-order logic statement is true about $f$:

$$\forall b \in B. \exists a \in A. f(a) = b$$

(“For every output, there's an input that produces it.”)

• A function with this property is called a **surjection**.

• How does this compare to our first rule of functions?
**Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.
Surjective Functions

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Proof:

What does it mean for \( f \) to be surjective?

\[ \forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y \]

Therefore, we'll choose an arbitrary \( y \in \mathbb{R} \), then prove that there is some \( x \in \mathbb{R} \) where \( f(x) = y \).
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**Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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$$\forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$. 

Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$. Let $x = 2y$. Then we see that $f(x) = f(2y) = 2y/2 = y$. So $f(x) = y$, as required. ■
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**Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

**Proof:**

Let $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y/2$. Then we see that $f(x) = f(y/2) = 2y/2 = y$.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.
Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Question: What do we need to do to prove that $g$ is not surjective? Try taking the definition of surjectivity and then negating it.

Respond at pollev.com/zhenglian740
**Theorem:** Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for $g$ to be surjective?

\[ \forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n \]
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$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. g(m) \neq n$$

Therefore, we need to find a natural number $n$ where, regardless of which $m$ we pick, we have $g(m) \neq n$. 


**Theorem:** Let \( g : \mathbb{N} \to \mathbb{N} \) be defined as \( g(n) = 2n \). Then \( g(x) \) is not surjective.

**Proof:**
Surjective Functions

**Theorem:** Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

**Proof:** Let $n = 137$. 

Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required. ■
**Surjective Functions**

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**Proof:** Let $n = 137$.

---

Our overall goal is to prove

$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. \ g(m) \neq n.$$  

We just made our choice of $n$. Therefore, we need to prove

$$\forall m \in \mathbb{N}. \ g(m) \neq n.$$  

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$. 
**Theorem:** Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

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A Proof About Birds
**Theorem:** If all birds can fly, then all herons can fly.
**Theorem:** If all birds can fly, then all herons can fly.

Given the predicates

\[
\text{Bird}(b), \text{ which says } b \text{ is a bird;}
\]
\[
\text{Heron}(h), \text{ which says } h \text{ is a heron; and}
\]
\[
\text{CanFly}(x), \text{ which says } x \text{ can fly},
\]

translate the theorem into first-order logic.
**Theorem:** If all birds can fly, then all herons can fly.

Given the predicates

- \(\text{Bird}(b)\), which says \(b\) is a bird;
- \(\text{Heron}(h)\), which says \(h\) is a heron; and
- \(\text{CanFly}(x)\), which says \(x\) can fly,

translate the theorem into first-order logic.

\[
(\forall b. (\text{Bird}(b) \rightarrow \text{CanFly}(b))) \rightarrow (\forall h. (\text{Heron}(h) \rightarrow \text{CanFly}(h)))
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| A → B  | Assume \( A \) is true, then prove \( B \) is true. |
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\[
(\forall b. (\text{Bird}(b) \rightarrow \text{CanFly}(b))) \rightarrow (\forall h. (\text{Heron}(h) \rightarrow \text{CanFly}(h)))
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$(\forall b. \ (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. \ (Heron(h) \rightarrow CanFly(h)))$

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Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird \( b \).
2. Consider an arbitrary heron \( h \).

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**Theorem:** If all birds can fly, then all herons can fly.

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**Theorem:** If all birds can fly, then all herons can fly.

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**Theorem:** If all birds can fly, then all herons can fly.

**Proof:** Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary bird $b$. Since $b$ is a bird, $b$ can fly. [*and now we’re stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!*]
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2. Consider an arbitrary heron \( h \).
**Theorem:** If all birds can fly, then all herons can fly.

**Proof:** Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary citril \( c \). We will show that \( c \) can fly. To do so, note that since \( c \) is a citril we know \( c \) is a bird. Therefore, by our earlier assumption, \( c \) can fly. ■

Which makes more sense as the next step in this proof?

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2. Consider an arbitrary heron \( h \).
**Theorem:** If all birds can fly, then all herons can fly.

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**Theorem:** If all birds can fly, then all herons can fly.

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Consider an arbitrary heron $h$. We will show that $h$ can fly. To do so, note that since $h$ is a heron we know $h$ is a bird. Therefore, by our earlier assumption, $h$ can fly. ■

$$(\forall b. \ (\text{Bird}(b) \rightarrow \text{CanFly}(b))) \rightarrow (\forall h. \ (\text{Heron}(h) \rightarrow \text{CanFly}(h)))$$

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$$(\forall b. (\text{Bird}(b) \rightarrow \text{CanFly}(b))) \rightarrow (\forall h. (\text{Heron}(h) \rightarrow \text{CanFly}(h)))$$

- We never introduce a variable $b$.
- We introduce a variable $h$ almost immediately.
Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we **assumed** all birds can fly.
  - Here, we **proved** all herons can fly.
- Statements behave differently based on whether you’re assuming or proving them.

\[(\forall b. \ (\text{Bird}(b) \rightarrow \text{CanFly}(b))) \rightarrow (\forall h. \ (\text{Heron}(h) \rightarrow \text{CanFly}(h)))\]

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---

**Proving vs. Assuming**

- To **prove** the universally-quantified statement

  \[
  \forall x. P(x)
  \]

  we introduce a new variable \( x \) representing some arbitrarily-chosen value.

- Then, we prove that \( P(x) \) is true for that variable \( x \).

- That’s why we introduced a variable \( h \) in this proof representing a heron.
Proving vs. Assuming

• If we *assume* the statement
  \[ \forall x. P(x) \]
  we *do not* introduce a variable \( x \).

• Rather, if we find a relevant value \( z \) somewhere else in the proof, we can conclude that \( P(z) \) is true.

• That’s why we didn’t introduce a variable \( b \) in our proof, and why we concluded that \( h \), our heron, can fly.

\[(\forall b. (\text{Bird}(b) \rightarrow \text{CanFly}(b)))) \rightarrow (\forall h. (\text{Heron}(h) \rightarrow \text{CanFly}(h))))\]

We never introduce a variable \( b \).

We introduce a variable \( h \) almost immediately.
To *prove* that this is true...

<p>| ( \forall x. \ A ) | Have the reader pick an arbitrary ( x ). We then prove ( A ) is true for that choice of ( x ). |
| ( \exists x. \ A ) | Find an ( x ) where ( A ) is true. Then prove that ( A ) is true for that specific choice of ( x ). |
| ( A \rightarrow B ) | Assume ( A ) is true, then prove ( B ) is true. |
| ( A \land B ) | Prove ( A ). Then prove ( B ). |
| ( A \lor B ) | Either prove ( \neg A \rightarrow B ) or prove ( \neg B \rightarrow A ). <em>(Why does this work?)</em> |
| ( A \leftrightarrow B ) | Prove ( A \rightarrow B ) and ( B \rightarrow A ). |
| ( \neg A ) | Simplify the negation, then consult this table on the result. |</p>
<table>
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| **∃x. A** | Find an x where A is true. Then prove that A is true for that specific choice of x. | Introduce a variable x into your proof that has property A. |

| **A → B** | Assume A is true, then prove B is true. | Initially, **do nothing**. Once you know A is true, you can conclude B is also true. |

| **A ∧ B** | Prove A. Then prove B. | Assume A. Then assume B. |

| **A ∨ B** | Either prove ¬A → B or prove ¬B → A. | *(Why does this work?)* |

| **A ↔ B** | Prove A → B and B → A. | |

| **¬A** | Simplify the negation, then consult this table on the result. | 
| **∀x. A** | Have the reader pick an arbitrary \( x \). We then prove \( A \) is true for that choice of \( x \). | Initially, **do nothing**. Once you find a \( z \) through other means, you can state it has property \( A \). |
| **∃x. A** | Find an \( x \) where \( A \) is true. Then prove that \( A \) is true for that specific choice of \( x \). | Introduce a variable \( x \) into your proof that has property \( A \). |
| **A → B** | Assume \( A \) is true, then prove \( B \) is true. | Initially, **do nothing**. Once you know \( A \) is true, you can conclude \( B \) is also true. |
| **A ∧ B** | Prove \( A \). Then prove \( B \). | Assume \( A \). Then assume \( B \). |
| **A ∨ B** | Either prove \( \neg A → B \) or prove \( \neg B → A \). *(Why does this work?)* | Consider two cases. Case 1: \( A \) is true. Case 2: \( B \) is true. |
| **A ↔ B** | Prove \( A → B \) and \( B → A \). | Assume \( A → B \) and \( B → A \). |
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Recap from Today

- A *function* takes in an element of a *domain* and maps it to an element of a *codomain*. Functions must be deterministic.

- Definitions are often given in first-order logic, and the structure of a first-order logic statement dictates the structure of a proof.

- *Involution*, *injection*, and *surjection* are specific classes of functions that have nice properties.
Next Time

- **Connecting Function Types**
  - Involutions, injections, and surjections are related to one another. How?

- **Function Composition**
  - Sequencing functions together.