Binary Relations
Part One
Outline for Today

• **Binary Relations**
  • Reasoning about connections between objects.

• **Equivalence Relations**
  • Reasoning about clusters.

• **A Fundamental Theorem**
  • How do we know we have the “right” definition for something?
In CS103, you've seen examples of relationships:

- between sets:
  - $A \subseteq B$
- between numbers:
  - $x < y \quad x \equiv_k y \quad x \leq y$
- between people:
  - $p$ loves $q$

Since these relations focus on connections between two objects, they are called **binary relations**.

- The “binary” here means “pertaining to two things,” not “made of zeros and ones.”
What exactly is a binary relation?
10 < 12
7 ≡₃ 10
$6 \equiv_3 11$
Binary Relations

• A *binary relation over a set* \( A \) is a predicate \( R \) that can be applied to pairs of elements drawn from \( A \).

• If \( R \) is a binary relation over \( A \) and it holds for the pair \((a, b)\), we write \( aRb \).
  • For example: \( 3 = 3 \), \( 5 < 7 \), and \( \emptyset \subseteq \mathbb{N} \).

• If \( R \) is a binary relation over \( A \) and it does not hold for the pair \((a, b)\), we write \( a\neg Rb \).
  • For example: \( 4 \neq 3 \), \( 4 \not< 3 \), and \( \mathbb{N} \not\subseteq \emptyset \).
Properties of Relations

• Generally speaking, if $R$ is a binary relation over a set $A$, the order of the operands is significant.
  • For example, $3 < 5$, but $5 \not< 3$.
  • In some relations order is irrelevant; more on that later.

• Relations are always defined relative to some underlying set.
  • It's not meaningful to ask whether $\odot \subseteq 15$, for example, since $\subseteq$ is defined over sets, not arbitrary objects.
We can visualize a binary relation $R$ over a set $A$ by drawing the elements of $A$ and drawing a line between an element $a$ and an element $b$ if $aRb$ is true.

Example: the relation $a \mid b$ (meaning “$a$ divides $b$”) over the set $\{1, 2, 3, 4\}$ looks like this:
Visualizing Relations

• We can visualize a binary relation $R$ over a set $A$ by drawing the elements of $A$ and drawing a line between an element $a$ and an element $b$ if $aRb$ is true.

• Example: the relation $a \neq b$ over the set $\{1, 2, 3, 4\}$ looks like this:
Visualizing Relations

- We can visualize a binary relation $R$ over a set $A$ by drawing the elements of $A$ and drawing a line between an element $a$ and an element $b$ if $aRb$ is true.

- Example: the relation $a = b$ over the set $\{1, 2, 3, 4\}$ looks like this:
Visualizing Relations

- We can visualize a binary relation $R$ over a set $A$ by drawing the elements of $A$ and drawing a line between an element $a$ and an element $b$ if $aRb$ is true.

- Example: below is some relation over $\{1, 2, 3, 4\}$ that's a totally valid relation even though there doesn't appear to be a simple unifying rule.
Capturing Structure
Capturing Structure

• Binary relations are an excellent way for capturing certain structures that appear in computer science.

• Today, we'll look at one of them (partitions), and next time we'll see another (prerequisites).

• Along the way, we'll explore how to write proofs about definitions given in first-order logic.
Partitions
Partitions

• A *partition of a set* is a way of splitting the set into disjoint, nonempty subsets so that every element belongs to exactly one subset.
  • Two sets are *disjoint* if their intersection is the empty set; formally, sets $S$ and $T$ are disjoint if $S \cap T = \emptyset$.

• Intuitively, a partition of a set breaks the set apart into smaller pieces.

• There doesn't have to be any rhyme or reason to what those pieces are, though often there is one.
Partitions and Clustering

• If you have a set of data, you can often learn something from the data by finding a “good” partition of that data and inspecting the partitions.
  • Usually, the term *clustering* is used in data analysis rather than *partitioning*.
• Interested to learn more? Take CS161 or CS246!
What's the connection between partitions and binary relations?
$aRa$

$aRb$ $\rightarrow$ $bRa$

$aRb \land bRc$ $\rightarrow$ $aRc$
\[
\forall a \in A. \ aRa
\]

\[
\forall a \in A. \ \forall b \in A. \ (aRb \to bRa)
\]

\[
\forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \to aRc)
\]
\[ \forall a \in A. \; aRa \]

\[ \forall a \in A. \; \forall b \in A. \; (aRb \rightarrow bRa) \]

\[ \forall a \in A. \; \forall b \in A. \; \forall c \in A. \; (aRb \land bRc \rightarrow aRc) \]
Reflexivity

- Some relations always hold from any element to itself.
- Examples:
  - \( x = x \) for any \( x \).
  - \( A \subseteq A \) for any set \( A \).
  - \( x \equiv_k x \) for any \( x \).
- Relations of this sort are called **reflexive**.
- Formally speaking, a binary relation \( R \) over a set \( A \) is reflexive if the following is true:

  \[
  \forall a \in A. \ aRa
  \]

  ("Every element is related to itself.")
Reflexivity Visualized

∀a ∈ A. aRa

(“Every element is related to itself.”)
Is This Relation Reflexive?

∀a ∈ A. aRa
(“Every element is related to itself.”)
Is This Relation Reflexive?

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∀a ∈ A. aRa

∀a ∈ A. ∀b ∈ A. (aRb → bRa)

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)
∀a ∈ A. aRa

∀a ∈ A. ∀b ∈ A. (aRb → bRa)

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)
Symmetry

• In some relations, the relative order of the objects doesn't matter.
• Examples:
  • If $x = y$, then $y = x$.
  • If $x \equiv_k y$, then $y \equiv_k x$.
• These relations are called symmetric.
• Formally: a binary relation $R$ over a set $A$ is called symmetric if the following first-order statement is true about $R$:

\[ \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \]

("If $a$ is related to $b$, then $b$ is related to $a$."")
∀a ∈ A. ∀b ∈ A. (aRb → bRa)
(“If a is related to b, then b is related to a.”)
Is This Relation Symmetric?

\[ \forall a \in A. \forall b \in A. (aRb \rightarrow bRa) \]

("If a is related to b, then b is related to a.")
Is This Relation Symmetric?

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∀a ∈ A. aRa

∀a ∈ A. ∀b ∈ A. (aRb → bRa)

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)
\[ \forall a \in A. \ aRa \]

\[ \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \]

\[ \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc) \]
Transitivity

• Many relations can be chained together.
• Examples:
  • If $x = y$ and $y = z$, then $x = z$.
  • If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  • If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
• These relations are called *transitive*.
• A binary relation $R$ over a set $A$ is called *transitive* if the following first-order statement is true about $R$:

  $$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

  (“Whenever $a$ is related to $b$ and $b$ is related to $c$, we know $a$ is related to $c$.”)
∀ a ∈ A. ∀ b ∈ A. ∀ c ∈ A. (aRb ∧ bRc → aRc)

(“Whenever a is related to b and b is related to c, we know a is related to c.”)
Is This Relation Transitive?

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)

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(“Whenever a is related to b and b is related to c, we know a is related to c.”)
Equivalence Relations

• An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

• Some examples:
  • $x = y$
  • $x \equiv_k y$
  • $x$ has the same color as $y$
  • $x$ has the same shape as $y$. 
Binary relations give us a *common language* to describe *common structures*. 
Equivalence Relations

- Most modern programming languages include some sort of hash table data structure.
  - Java: HashMap
  - C++: std::unordered_map
  - Python: dict
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...
Equivalence Relations

“The equals method implements an equivalence relation on non-null object references:

- It is reflexive: for any non-null reference value $x$, $x.equals(x)$ should return true.
- It is symmetric: for any non-null reference values $x$ and $y$, $x.equals(y)$ should return true if and only if $y.equals(x)$ returns true.
- It is transitive: for any non-null reference values $x$, $y$, and $z$, if $x.equals(y)$ returns true and $y.equals(z)$ returns true, then $x.equals(z)$ should return true.”

Java 8 Documentation
Equivalence Relations

“The equals method implements an equivalence relation on non-null object references:

- **It is reflexive**: for any non-null reference value x, x.equals(x) should return true.
- **It is symmetric**: for any non-null reference values x and y, x.equals(y) should return true if and only if y.equals(x) returns true.
- **It is transitive**: for any non-null reference values x, y, and z, if x.equals(y) returns true and y.equals(z) returns true, then x.equals(z) should return true.”
Equivalence Relations

“Each unordered associative container is parameterized by Key, by a function object type Hash that meets the Hash requirements (17.6.3.4) and acts as a hash function for argument values of type Key, and by a binary predicate Pred that induces an equivalence relation on values of type Key. Additionally, unordered_map and unordered_multimap associate an arbitrary mapped type T with the Key.”

C++14 ISO Spec, §23.2.5/3
Equivalence Relations

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Time-Out for Announcements!
Problem Set Two

• The Problem Set Two checkpoint problem was due at 3:00PM today.
  • We'll get back to you with feedback by Wednesday.
  • Solutions are available – please read over them! The checkpoint problem covers a lot of interesting nuances of first-order logic and you should be absolutely certain you completely understand all the answers.

• The remaining problems are due on Friday.
  • Please feel free to stop by office hours with questions!
Problem Set One Solutions

• Problem Set One solutions are now available.
  
  • Please read over the solutions. Each problem was chosen for a reason, and it's important to both see one possible solution and the motivation behind the problem.
  
  • Make sure you understand the solutions. If you don't understand the solutions, please come talk to us and ask us questions. That's how you learn!
  
• We'll try to get graded problem sets back by Wednesday.
Back to CS103!
Equivalence Relation Proofs

• Let's suppose you've found a binary relation $R$ over a set $A$ and want to prove that it's an equivalence relation.

• How exactly would you go about doing this?
An Example Relation

- Consider the binary relation \( \sim \) defined over the set \( \mathbb{Z} \):
  \[
  a \sim b \quad \text{if} \quad a+b \text{ is even}
  \]
- Some examples:
  \[
  0 \sim 4 \quad 1 \sim 9 \quad 2 \sim 6 \quad 5 \sim 5
  \]
- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

\[
 a \sim b \quad \text{if} \quad \text{some property of } a \text{ and } b \text{ holds}
\]

This is the general template for defining a relation. Although we're using “if” rather than “iff” here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with “if” rather than “iff.” Check the “Mathematical Vocabulary” handout for details.
What properties must $\sim$ have to be an equivalence relation?

**Reflexivity**

**Symmetry**

**Transitivity**

Let's prove each property independently.
Lemma 1: The binary relation \( \sim \) is reflexive.

\[ a \sim b \quad \text{if} \quad a + b \text{ is even} \]
Lemma 1: The binary relation ≃ is reflexive.

Proof:

\[ a \sim b \text{ if } a+b \text{ is even} \]
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Proof:

What is the formal definition of reflexivity?
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Proof:

What is the formal definition of reflexivity?

$\forall a \in \mathbb{Z}. \ a \sim a$
\( a \sim b \quad \text{if} \quad a + b \text{ is even} \)

**Lemma 1:** The binary relation \( \sim \) is reflexive.

**Proof:**

What is the formal definition of reflexivity?

\[ \forall a \in \mathbb{Z}. \ a \sim a \]

Therefore, we'll choose an arbitrary integer \( a \), then go prove that \( a \sim a \).
\( a \sim b \quad \text{if} \quad a + b \text{ is even} \)

**Lemma 1:** The binary relation \( \sim \) is reflexive.

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\( \forall a \in \mathbb{Z}. \ a \sim a \)

Therefore, we’ll choose an arbitrary integer \( a \), then go prove that \( a \sim a \).
Lemma 1: The binary relation ~ is reflexive.

Proof:

\[ a \sim b \text{ if } a + b \text{ is even} \]

What is the formal definition of reflexivity?

\[ \forall a \in \mathbb{Z}. \quad a \sim a \]

Therefore, we'll choose an arbitrary integer \( a \), then go prove that \( a \sim a \).
Lemma 1: The binary relation ~ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. 

If $a + b$ is even, then $a \sim b$. 

$a \sim b$ if $a + b$ is even
Lemma 1: The binary relation \( \sim \) is reflexive.

Proof: Consider an arbitrary \( a \in \mathbb{Z} \). We need to prove that \( a \sim a \). From the definition of the \( \sim \) relation, this means that we need to prove that \( a+a \) is even. To see this, notice that \( a+a = 2a \), so the sum \( a+a \) can be written as \( 2k \) for some integer \( k \) (namely, \( a \)), so \( a+a \) is even. Therefore, \( a \sim a \) holds, as required. \(\square\)
Lemma 1: The binary relation $\sim$ is reflexive.

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Lemma 2: The binary relation $\sim$ is symmetric.

$a \sim b$ if $a + b$ is even
Lemma 2: The binary relation $\sim$ is symmetric.

Proof:

$a \sim b$ if $a + b$ is even
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Proof:

What is the formal definition of symmetry?
Lemma 2: The binary relation $\sim$ is symmetric.

Proof:

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \ (a \sim b \rightarrow b \sim a)$$
\[ a \sim b \quad \text{if} \quad a+b \text{ is even} \]

**Lemma 2:** The binary relation \( \sim \) is symmetric.

**Proof:**

What is the formal definition of symmetry?

\[ \forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \ (a \sim b \rightarrow b \sim a) \]

Therefore, we'll choose arbitrary integers \( a \) and \( b \) where \( a \sim b \), then prove that \( b \sim a \).
Lemma 2: The binary relation ~ is symmetric.

Proof:

\[ a \sim b \text{ if } a+b \text{ is even} \]

\begin{align*}
\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. (a \sim b \rightarrow b \sim a)
\end{align*}

Therefore, we'll choose arbitrary integers \(a\) and \(b\) where \(a \sim b\), then prove that \(b \sim a\).
\[ a \sim b \text{ if } a + b \text{ is even} \]

**Lemma 2:** The binary relation \( \sim \) is symmetric.

**Proof:**

What is the formal definition of symmetry?

\[
\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. (a \sim b \rightarrow b \sim a)
\]

Therefore, we'll choose arbitrary integers \( a \) and \( b \) where \( a \sim b \), then prove that \( b \sim a \).
Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers $a$ and $b$ where $a\sim b$. We need to show that $b\sim a$. If $a+b$ is even, then $b+a$ is also even, which means $b\sim a$. ■
Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers $a$ and $b$ where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a + b$ is even.
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Proof: Consider any integers \( a \) and \( b \) where \( a \sim b \). We need to show that \( b \sim a \).

Since \( a \sim b \), we know that \( a+b \) is even. Because \( a+b = b+a \), this means that \( b+a \) is even.
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Since $a \sim b$, we know that $a + b$ is even. Because $a + b = b + a$, this means that $b + a$ is even. Since $b + a$ is even, we know that $b \sim a$, as required.
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Lemma 3: The binary relation \( \sim \) is transitive.

Proof: Consider arbitrary integers \( a \), \( b \), and \( c \) where \( a \sim b \) and \( b \sim c \). We need to prove that \( a \sim c \), meaning that we need to show that \( a + c \) is even.

Since \( a \sim b \) and \( b \sim c \), we know that \( a \sim b \) and \( b \sim c \) are even. This means there are integers \( k \) and \( m \) where \( a + b = 2k \) and \( b + c = 2m \). Notice that \((a + b) + (b + c) = 2k + 2m\).

Rearranging, we see that \( a + c + 2b = 2k + 2m \), so \( a + c = 2k + 2m - 2b = 2(k + m - b) \).

So there is an integer \( r \), namely \( k + m - b \), such that \( a + c = 2r \). Thus \( a + c \) is even, so \( a \sim c \), as required. \( \blacksquare \)
Lemma 3: The binary relation ~ is transitive.

Proof: $a \sim b$ if $a+b$ is even
Lemma 3: The binary relation \( \sim \) is transitive.

Proof:

\[ a \sim b \text{ if } a + b \text{ is even} \]

What is the formal definition of transitivity?
Lemma 3: The binary relation \( \sim \) is transitive.

**Proof:**

What is the formal definition of transitivity?

\[
\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \forall c \in \mathbb{Z}. \ (a \sim b \land b \sim c \rightarrow a \sim c)
\]
Lemma 3: The binary relation ~ is transitive.

Proof:

What is the formal definition of transitivity?

∀a ∈ ℤ. ∀b ∈ ℤ. ∀c ∈ ℤ. (a ~ b ∧ b ~ c → a ~ c)

Therefore, we’ll choose arbitrary integers a, b, and c where a ~ b and b ~ c, then prove that a ~ c.
Lemma 3: The binary relation \( \sim \) is transitive.

Proof: Consider arbitrary integers \( a, b \) and \( c \) where \( a \sim b \) and \( b \sim c \).
Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a + c$ is even.
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Proof: Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a + c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a + b$ and $b + c$ are even.
Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers $k$ and $m$ where $a+b = 2k$ and $b+c = 2m$. 

\[a \sim b \quad \text{if} \quad a+b \text{ is even}\]
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**Lemma 3:** The binary relation \( \sim \) is transitive.

**Proof:** Consider arbitrary integers \( a, b \) and \( c \) where \( a \sim b \) and \( b \sim c \). We need to prove that \( a \sim c \), meaning that we need to show that \( a+c \) is even.

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\[(a+b) + (b+c) = 2k + 2m.\]
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$$(a+b) + (b+c) = 2k + 2m.$$  

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so $a+c$ is even, so $a \sim c$, as required. ■
\[ a \sim b \quad \text{if} \quad a + b \text{ is even} \]

**Lemma 3:** The binary relation \( \sim \) is transitive.

**Proof:** Consider arbitrary integers \( a, b \) and \( c \) where \( a \sim b \) and \( b \sim c \). We need to prove that \( a \sim c \), meaning that we need to show that \( a + c \) is even.

Since \( a \sim b \) and \( b \sim c \), we know that \( a + b \) and \( b + c \) are even. This means there are integers \( k \) and \( m \) where \( a + b = 2k \) and \( b + c = 2m \). Notice that

\[(a+b) + (b+c) = 2k + 2m.\]

Rearranging, we see that

\[ a + c + 2b = 2k + 2m, \]

so

\[ a + c = 2k + 2m - 2b = 2(k + m - b). \]
Lemma 3: The binary relation ∼ is transitive.

Proof: Consider arbitrary integers \( a, b \) and \( c \) where \( a \sim b \) and \( b \sim c \). We need to prove that \( a \sim c \), meaning that we need to show that \( a + c \) is even.

Since \( a \sim b \) and \( b \sim c \), we know that \( a + b \) and \( b + c \) are even. This means there are integers \( k \) and \( m \) where \( a + b = 2k \) and \( b + c = 2m \). Notice that

\[
(a + b) + (b + c) = 2k + 2m.
\]

Rearranging, we see that

\[
a + c + 2b = 2k + 2m,
\]

so

\[
a + c = 2k + 2m - 2b = 2(k + m - b).
\]

So there is an integer \( r \), namely \( k + m - b \), such that \( a + c = 2r \). Thus \( a + c \) is even, so \( a \sim c \), as required.
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We know what equivalence relations are.
So what do equivalence relations do?
Properties of Equivalence Relations
\( xRy \) if \( x \) and \( y \) have the same shape
$xRy$ if $x$ and $y$ have the same shape
\[ x T y \quad \text{if} \quad x \text{ is the same color as } y \]
$xTy$ if $x$ is the same color as $y$
Equivalence Classes

- Given an equivalence relation $R$ over a set $A$, for any $x \in A$, the **equivalence class of $x$** is the set
  $$[x]_R = \{ y \in A \mid xRy \}$$
- $[x]_R$ is the set of all elements of $A$ that are related to $x$ by relation $R$.
- For example, consider the $\equiv_3$ relation over $\mathbb{N}$. Then
  - $[0]_{\equiv_3} = \{0, 3, 6, 9, 12, 15, 18, \ldots \}$
  - $[1]_{\equiv_3} = \{1, 4, 7, 10, 13, 16, 19, \ldots \}$
  - $[2]_{\equiv_3} = \{2, 5, 8, 11, 14, 17, 20, \ldots \}$
  - $[3]_{\equiv_3} = \{0, 3, 6, 9, 12, 15, 18, \ldots \}$
  Notice that $[0]_{\equiv_3} = [3]_{\equiv_3}$. These are *literally* the same set, so they're just different names for the same thing.
The Fundamental Theorem of Equivalence Relations: Let $R$ be an equivalence relation over a set $A$. Then every element $a \in A$ belongs to exactly one equivalence class of $R$. 