Binary Relations
Part II
Outline for Today

- **Finish from Last Time**
  - Pts. 2, 3 of our proof that \( \sim \) is an equivalence relation

- **Properties of Equivalence Relations**
  - What’s so special about those three rules?

- **Cyclic Property**
  - How it relates to our other three properties, and equivalence relations
Finish from Last Time
\[ \forall a \in A. \ aRa \]
\[ \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \]
\[ \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc) \]
An Example Relation

• Consider the binary relation \( \sim \) defined over the set \( \mathbb{Z} \):
  
  \[ a \sim b \text{ if } a+b \text{ is even} \]

• Some examples:
  
  \[ 0 \sim 4 \quad 1 \sim 9 \quad 2 \sim 6 \quad 5 \sim 5 \]

• Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

\[ a \sim b \text{ if } \text{some property of } a \text{ and } b \text{ holds} \]

*This is the general template for defining a relation.* Although we're using “if” rather than “iff” here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with “if” rather than “iff.” Check the “Mathematical Vocabulary” handout for details.
Lemma 1: The binary relation \( \sim \) is reflexive.

Proof: Consider an arbitrary \( a \in \mathbb{Z} \). We need to prove that \( a \sim a \). From the definition of the \( \sim \) relation, this means that we need to prove that \( a + a \) is even.

To see this, notice that \( a + a = 2a \), so the sum \( a + a \) can be written as \( 2k \) for some integer \( k \) (namely, \( a \)), so \( a + a \) is even. Therefore, \( a \sim a \) holds, as required. ■
Lemma 2: The binary relation \( \sim \) is symmetric.

\[
a \sim b \text{ if } a + b \text{ is even}
\]
Lemma 2: The binary relation \( \sim \) is symmetric.

Which of the following works best as the opening of this proof?

A. Consider any integers \( a \) and \( b \). We will prove \( a \sim b \) and \( b \sim a \).
B. Pick \( \forall a \in \mathbb{Z} \) and \( \forall b \in \mathbb{Z} \). We will prove \( a \sim b \rightarrow b \sim a \).
C. Consider any integers \( a \) and \( b \) where \( a \sim b \) and \( b \sim a \).
D. Consider any integer \( a \) where \( a \sim a \).
E. The relation \( \sim \) is symmetric if for any \( a, b \in \mathbb{Z} \), we have \( a \sim b \rightarrow b \sim a \).
F. **Consider any integers \( a \) and \( b \) where \( a \sim b \). We will prove \( b \sim a \).**

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then A, B, C, D, E, or F.
$a \sim b$ if $a + b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof:
Lemma 2: The binary relation \( \sim \) is symmetric.

Proof:

What is the formal definition of symmetry?
Lemma 2: The binary relation \( \sim \) is symmetric.

Proof:

\[
\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. (a \sim b \rightarrow b \sim a)
\]
Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

∀a ∈ ℤ. ∀b ∈ ℤ. (a ~ b → b ~ a)

Therefore, we'll choose arbitrary integers \( a \) and \( b \) where \( a \sim b \), then prove that \( b \sim a \).
Lemma 2: The binary relation \( \sim \) is symmetric.

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Lemma 2: The binary relation \( \sim \) is symmetric.

Proof: Consider any integers \( a \) and \( b \) where \( a \sim b \). We need to show that \( b \sim a \).

Since \( a \sim b \), we know that \( a + b \) is even.
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**Proof:** Consider any integers \( a \) and \( b \) where \( a \sim b \). We need to show that \( b \sim a \).

Since \( a \sim b \), we know that \( a + b \) is even. Because \( a + b = b + a \), this means that \( b + a \) is even.
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Since $a \sim b$, we know that $a + b$ is even. Because $a + b = b + a$, this means that $b + a$ is even. Since $b + a$ is even, we know that $b \sim a$, as required.
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∀a ∈ A. aRa

∀a ∈ A. ∀b ∈ A. (aRb → bRa)

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)
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Lemma 3: The binary relation ~ is transitive.
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Proof: Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a + c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a \sim b$ and $b \sim c$ are even. This means there are integers $k$ and $m$ where $a + b = 2k$ and $b + c = 2m$. Notice that $(a + b) + (b + c) = 2k + 2m$.

Rearranging, we see that $a + c + 2b = 2k + 2m$, so $a + c = 2k + 2m - 2b = 2(k + m - b)$.

So there is an integer $r$, namely $k + m - b$, such that $a + c = 2r$. Thus $a + c$ is even, so $a \sim c$, as required. ■

Which of the following works best as the introduction to this proof?

A. Pick an arbitrary $a$ and $b$ from $A$ where $a \sim b$. We’ll prove $b \sim a$.
B. Consider any $a$, $b$, $c \in A$ where $a \sim b$, $b \sim c$, and $a \sim c$.
C. Choose an $a$, $b$, $c \in A$. We will prove $a \sim b$, $b \sim c$, and $a \sim c$.
D. Take any $a$, $b$, $c \in A$ where $a \sim b$ and $b \sim c$; we’ll prove $a \sim c$.

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then A, B, C, or D.
$a \sim b$ if $a+b$ is even

**Lemma 3**: The binary relation $\sim$ is transitive.

**Proof**: 

Consider arbitrary integers $a$, $b$, and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a + c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a \sim b$ and $b \sim c$ are even. This means there are integers $k$ and $m$ where $a+b = 2k$ and $b+c = 2m$. Notice that $(a+b) + (b+c) = 2k + 2m$.

Rearranging, we see that $a+c + 2b = 2k + 2m$, so $a+c = 2k + 2m - 2b = 2(k+m-b)$. So there is an integer $r$, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required. ■
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Lemma 3: The binary relation \( \sim \) is transitive.

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What is the formal definition of transitivity?

\[
\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \forall c \in \mathbb{Z}. \ (a \sim b \land b \sim c \rightarrow a \sim c)
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Lemma 3: The binary relation ~ is transitive.

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What is the formal definition of transitivity?

∀a ∈ ℤ. ∀b ∈ ℤ. ∀c ∈ ℤ. (a ~ b ∧ b ~ c → a ~ c)

Therefore, we'll choose arbitrary integers a, b, and c where a ~ b and b ~ c, then prove that a ~ c.
\[ a \sim b \text{ if } a + b \text{ is even} \]

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Since $a \sim b$ and $b \sim c$, we know that $a + b$ and $b + c$ are even.
\[ a \sim b \quad \text{if} \quad a+b \text{ is even} \]

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Since \( a \sim b \) and \( b \sim c \), we know that \( a + b \) and \( b + c \) are even. This means there are integers \( k \) and \( m \) where \( a + b = 2k \) and \( b + c = 2m \).
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Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers $k$ and $m$ where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$
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\[(a+b) + (b+c) = 2k + 2m.\]

Rearranging, we see that

\[a + c + 2b = 2k + 2m,\]
Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers \( a, b \) and \( c \) where \( a\sim b \) and \( b\sim c \). We need to prove that \( a\sim c \), meaning that we need to show that \( a+c \) is even.

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(a+b) + (b+c) = 2k + 2m.
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Rearranging, we see that

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a+c + 2b = 2k + 2m,
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so

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a+c = 2k + 2m - 2b = 2(k+m-b).
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So there is an integer $r$, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a\sim c$, as required.
If $a + b$ is even

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$$a + c = 2k + 2m - 2b = 2(k + m - b).$$

So there is an integer $r$, namely $k + m - b$, such that $a + c = 2r$. Thus $a + c$ is even, so $a \sim c$, as required. $\blacksquare$
\[ a \sim b \quad \text{if} \quad a + b \text{ is even} \]

**Lemma 3:** The binary relation \( \sim \) is transitive.

**Proof:** Consider arbitrary integers \( a, b \) and \( c \) where \( a \sim b \) and \( b \sim c \). We need to prove that \( a \sim c \), meaning that we need to show that \( a + c \) is even.

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\[
(a + b) + (b + c) = 2k + 2m.
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Rearranging, we see that

\[
a + c + 2b = 2k + 2m,
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so

\[
a + c = 2k + 2m - 2b.
\]

So there is an integer \( r \), namely \( k + m - b \), such that \( a + c = 2r \). Thus \( a + c \) is even.

The formal definition of transitivity is given in first-order logic, but this proof does not contain any first-order logic symbols!
First-Order Logic and Proofs

• First-order logic is an excellent tool for giving formal definitions to key terms.

• While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.

• Follow the example of these proofs:
  • Use the FOL definitions to determine what to assume and what to prove.
  • Write the proof in plain English using the conventions we set up in the first week of the class.

• *Please, please, please, please, please, please, please* internalize the contents of this slide!
\( \forall a \in A. \ aRa \)

\( \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \)

\( \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc) \)
Properties of Equivalence Relations
$xRy$ if $x$ and $y$ have the same shape
\( xTy \) if \( x \) and \( y \) have the same color
Equivalence Classes

- Given an equivalence relation $R$ over a set $A$, for any $x \in A$, the **equivalence class of $x$** is the set
  
  $$[x]_R = \{ y \in A \mid xRy \}$$

- Intuitively, the set $[x]_R$ contains all elements of $A$ that are related to $x$ by relation $R$. 
\[ x R y \quad \text{if} \quad x \text{ and } y \text{ have the same shape} \]
The Fundamental Theorem of Equivalence Relations: Let $R$ be an equivalence relation over a set $A$. Then every element $a \in A$ belongs to exactly one equivalence class of $R$. 
\[ xRy \quad \text{if} \quad x \text{ and } y \text{ have the same shape} \]
How’d We Get Here?

- We discovered equivalence relations by thinking about\textit{ partitions} of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- \textbf{Question:} What’s so special about these three rules?
The question we are asking the sage: “Are these two in the same equivalence class?”
aRb \land bRc \rightarrow cRa
∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → cRa)
∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → cRa)

A binary relation with this property is called \textit{cyclic}. 
Let $R$ be the relation depicted here. How many of the following claims are true?

- $R$ is reflexive.
- $R$ is symmetric.
- $R$ is transitive.
- $R$ is an equivalence relation.

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then 0, 1, 2, 3, or 4.

\[ \forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow cRa) \]
Theorem: A binary relation $R$ over a set $A$ is an equivalence relation if and only if it is reflexive and cyclic.
**Theorem:** A binary relation $R$ over a set $A$ is an equivalence relation if and only if it is reflexive and cyclic.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

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**What We’re Assuming**

- $R$ is an equivalence relation.
- $R$ is reflexive.
- $R$ is symmetric.
- $R$ is transitive.

**What We Need To Show**

- $R$ is reflexive.
- $R$ is cyclic.
- If $aRb$ and $bRc$, then $cRa$. 

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**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

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**What We Need To Show**

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```
  a  
  ↓
  b
  ↓
  c
```

Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

What We’re Assuming

$R$ is an equivalence relation.

- $R$ is reflexive.
- $R$ is symmetric.
- $R$ is transitive.

What We Need To Show

- If $aRb$ and $bRc$, then $cRa$. 

![Diagram showing the relationship between a, b, and c with arrows indicating the transitive property.](image)
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:**

Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds.

Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$.

Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove. ■
**Lemma 1**: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

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![Diagram showing reflexive and cyclic relations](attachment:diagram.png)
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Next, we'll prove that $R$ is transitive. Let $a, b,$ and $c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

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Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

When writing proofs about terms with first-order definitions, it’s critical to call back to those definitions!
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

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Notice how this setup mirrors the first-order definition of transitivity:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

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Refining Your Proofwriting

• When writing proofs about terms with formal definitions, you **must** call back to those definitions.
  • Use the first-order definition to see what you’ll assume and what you’ll need to prove.

• When writing proofs about terms with formal definitions, you **must not** include any first-order logic in your proofs.
  • Although you won’t use any FOL notation in your proofs, your proof implicitly calls back to the FOL definitions.

• You’ll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!
Next Time

- **Functions**
  - How do we model transformations in a mathematical sense?
- **Domains and Codomains**
  - Type theory meets mathematics!
- **Injections, Surjections, and Bijections**
  - Three special classes of functions.