Outline for Today

• *Recap from Last Time*
  • Where are we, again?

• *Properties of Equivalence Relations*
  • What’s so special about those three rules?

• *Strict Orders*
  • A different type of mathematical structure

• *Hasse Diagrams*
  • How to visualize rankings
Recap from Last Time
Binary Relations

- A *binary relation over a set* *A* is a predicate *R* that can be applied to pairs of elements drawn from *A*.

- If *R* is a binary relation over *A* and it holds for the pair (*a*, *b*), we write *aRb*.
  - For example: $3 = 3$, $5 < 7$, and $\emptyset \subseteq \mathbb{N}$.

- If *R* is a binary relation over *A* and it does not hold for the pair (*a*, *b*), we write $a \not R b$.
  - For example: $4 \neq 3$, $4 \not< 3$, and $\mathbb{N} \not\subseteq \emptyset$. 
Reflexivity

- Some relations always hold from any element to itself.
- Examples:
  - $x = x$ for any $x$.
  - $A \subseteq A$ for any set $A$.
  - $x \equiv_k x$ for any $x$.
- Relations of this sort are called reflexive.
- Formally speaking, a binary relation $R$ over a set $A$ is reflexive if the following first-order logic statement is true about $R$:

$$\forall a \in A. \ aRa$$

(“Every element is related to itself.”)
Reflexivity Visualized

∀\ a ∈ A. \ aRa

("Every element is related to itself.")
Symmetry

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $x = y$, then $y = x$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called **symmetric**.
- Formally: a binary relation $R$ over a set $A$ is called **symmetric** if the following first-order statement is true about $R$:

$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

(“If $a$ is related to $b$, then $b$ is related to $a$.”)
∀a ∈ A. ∀b ∈ A. (aRb → bRa)
(“If a is related to b, then b is related to a.”)
Transitivity

• Many relations can be chained together.
• Examples:
  • If $x = y$ and $y = z$, then $x = z$.
  • If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  • If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
• These relations are called transitive.
• A binary relation $R$ over a set $A$ is called transitive if the following first-order statement is true about $R$:
  \[
  \forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)
  \]
  ("Whenever $a$ is related to $b$ and $b$ is related to $c$, we know $a$ is related to $c$.")
∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)
("Whenever a is related to b and b is related to c, we know a is related to c.")
Equivalence Relations

- An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - $x$ has the same color as $y$
  - $x$ has the same shape as $y$. 
Equivalence Classes

• Given an equivalence relation $R$ over a set $A$, for any $x \in A$, the *equivalence class of $x$* is the set
  $$[x]_R = \{ y \in A \mid xRy \}$$

• $[x]_R$ is the set of all elements of $A$ that are related to $x$ by relation $R$.

• For example, consider the $\equiv_3$ relation over $\mathbb{N}$.
  Then
  • $[0]_{\equiv_3} = \{0, 3, 6, 9, 12, 15, 18, \ldots\}$
  • $[1]_{\equiv_3} = \{1, 4, 7, 10, 13, 16, 19, \ldots\}$
  • $[2]_{\equiv_3} = \{2, 5, 8, 11, 14, 17, 20, \ldots\}$
  • $[3]_{\equiv_3} = \{0, 3, 6, 9, 12, 15, 18, \ldots\}$

Notice that $[0]_{\equiv_3} = [3]_{\equiv_3}$. These are literally the same set, so they're just different names for the same thing.
The Fundamental Theorem of Equivalence Relations: Let $R$ be an equivalence relation over a set $A$. Then every element $a \in A$ belongs to exactly one equivalence class of $R$. 
New Stuff!
How’d We Get Here?

- We discovered equivalence relations by thinking about partitions of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- **Question**: What’s so special about these three rules?
aRb \land bRc \rightarrow cRa
\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow cRa)
A binary relation with this property is called cyclic.
∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → cRa)

Is this an equivalence relation?
**Theorem:** A binary relation $R$ over a set $A$ is an equivalence relation if and only if it is reflexive and cyclic.
**Theorem:** A binary relation $R$ over a set $A$ is an equivalence relation *if and only if* it is reflexive and cyclic.
**Lemma 1:** If \( R \) is an equivalence relation over a set \( A \), then \( R \) is reflexive and cyclic.

**Lemma 2:** If \( R \) is a binary relation over a set \( A \) that is reflexive and cyclic, then \( R \) is an equivalence relation.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is an equivalence relation.</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is cyclic.</td>
</tr>
<tr>
<td>$R$ is symmetric.</td>
<td></td>
</tr>
<tr>
<td>$R$ is transitive.</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is an equivalence relation.</td>
<td>• $R$ is reflexive.</td>
</tr>
<tr>
<td>• $R$ is reflexive.</td>
<td>$R$ is cyclic.</td>
</tr>
<tr>
<td>• $R$ is symmetric.</td>
<td></td>
</tr>
<tr>
<td>• $R$ is transitive.</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>- $R$ is an equivalence relation.</td>
<td>- $R$ is reflexive.</td>
</tr>
<tr>
<td>- $R$ is reflexive.</td>
<td>- $R$ is cyclic.</td>
</tr>
<tr>
<td>- $R$ is symmetric.</td>
<td></td>
</tr>
<tr>
<td>- $R$ is transitive.</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $R$ is an equivalence relation.</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td>• $R$ is reflexive.</td>
<td>$R$ is cyclic.</td>
</tr>
<tr>
<td>• $R$ is symmetric.</td>
<td>• If $arb$ and $brc$, then $cra$.</td>
</tr>
<tr>
<td>• $R$ is transitive.</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 1**: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**What We’re Assuming**
- $R$ is an equivalence relation.
- $R$ is reflexive.
- $R$ is symmetric.
- $R$ is transitive.

**What We Need To Show**
- If $arb$ and $brc$, then $cRa$. 

![Diagram](image)
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

What We’re Assuming

- $R$ is an equivalence relation.
- $R$ is reflexive.
- $R$ is symmetric.
- $R$ is transitive.

What We Need To Show

- If $arb$ and $brc$, then $cRa$.

Diagram:

- $a$ is connected to $b$.
- $b$ is connected to $c$.
- $c$ is connected to $a$. 
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**What We’re Assuming**

- $R$ is an equivalence relation.
  - $R$ is reflexive.
  - $R$ is symmetric.
  - $R$ is transitive.

**What We Need To Show**

- If $arb$ and $brc$, then $cRa$. 

![Diagram](image)
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:**
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. 

Since $R$ is reflexive, we have $aRa$ for all $a \in A$. 

Now, let $a, b, c \in A$ such that $aRb$ and $bRc$. 

Since $R$ is transitive, we have $aRc$. 

Then, since $R$ is symmetric, we have $cRa$, which is what we needed to prove.  

$\blacksquare$
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. 
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds.
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. 
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove.
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove. ■
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Proof:** Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove. ■
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic. Since $R$ is an equivalence relation, we know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove. ■
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic.

Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove. ■

Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That’s normal – it’s actually quite rare to see first-order logic in written proofs.
**Lemma 1:** If \( R \) is an equivalence relation over a set \( A \), then \( R \) is reflexive and cyclic.

**Lemma 2:** If \( R \) is a binary relation over a set \( A \) that is reflexive and cyclic, then \( R \) is an equivalence relation.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td></td>
<td>$R$ is transitive.</td>
</tr>
</tbody>
</table>
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is an equivalence relation.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td></td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td></td>
<td>$R$ is transitive.</td>
</tr>
</tbody>
</table>
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is an equivalence relation.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td></td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td></td>
<td>$R$ is transitive.</td>
</tr>
</tbody>
</table>
**Lemma 2:** If \( R \) is a binary relation over a set \( A \) that is reflexive and cyclic, then \( R \) is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>• ( R ) is reflexive.</td>
<td>( R ) is an equivalence relation.</td>
</tr>
<tr>
<td>• ( R ) is cyclic.</td>
<td>( R ) is reflexive.</td>
</tr>
<tr>
<td></td>
<td>• ( R ) is symmetric.</td>
</tr>
<tr>
<td></td>
<td>( R ) is transitive.</td>
</tr>
</tbody>
</table>
Lemma 2: If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

**What We’re Assuming**

- $R$ is reflexive.
- $R$ is cyclic.

**What We Need To Show**

- $R$ is symmetric.
- If $aRb$, then $bRa$.
Lemma 2: If \( R \) is a binary relation over a set \( A \) that is reflexive and cyclic, then \( R \) is an equivalence relation.

What We’re Assuming

- \( R \) is reflexive.
- \( \forall x \in A. \) \( xRx \)
- \( R \) is cyclic.
- \( xRy \wedge yRz \rightarrow zRx \)

What We Need To Show

- \( R \) is symmetric.
- If \( arb \), then \( bra \).
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td>$\forall x \in A. \ xRx$</td>
<td>If $arb$, then $bra$.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td></td>
</tr>
<tr>
<td>$xRy \land yRz \rightarrow zRx$</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 2:** If \( R \) is a binary relation over a set \( A \) that is reflexive and cyclic, then \( R \) is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) is reflexive.</td>
<td>( R ) is symmetric.</td>
</tr>
<tr>
<td>( \forall x \in A, xRx )</td>
<td>If ( arb ), then ( bra ).</td>
</tr>
<tr>
<td>( R ) is cyclic.</td>
<td>( )</td>
</tr>
<tr>
<td>( xRy \land yRz \rightarrow zRx )</td>
<td>( a) ( b)</td>
</tr>
</tbody>
</table>
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td>$\forall x \in A. \ xRx$</td>
<td>If $arb$, then $bra$.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td></td>
</tr>
<tr>
<td>$xRy \land yRz \rightarrow zRx$</td>
<td></td>
</tr>
</tbody>
</table>

![Diagram](image)
Lemma 2: If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

What We’re Assuming

- $R$ is reflexive.
- $\forall x \in A. \ xRx$
- $R$ is cyclic.
- $xRy \land yRz \rightarrow zRx$

What We Need To Show

- $R$ is symmetric.
- $\text{If } arb, \text{ then } bRa.$
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

**What We’re Assuming**

$R$ is reflexive.

$\forall x \in A. \ xRx$

$R$ is cyclic.

- $xRy \land yRz \rightarrow zRx$

**What We Need To Show**

- $R$ is symmetric.

- If $arb$, then $bra$. 
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $R$ is reflexive.</td>
<td>• $R$ is symmetric.</td>
</tr>
<tr>
<td>• $\forall x \in A. \ xRx$</td>
<td>• If $arb$, then $bra$.</td>
</tr>
<tr>
<td>• $R$ is cyclic.</td>
<td></td>
</tr>
<tr>
<td>• $xRy \land yRz \rightarrow zRx$</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td>$\forall x \in A. \ xRx$</td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td>$R$ is transitive.</td>
</tr>
<tr>
<td>$xRy \land yRz \rightarrow zRx$</td>
<td>$R$ is an equivalence relation.</td>
</tr>
</tbody>
</table>
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $R$ is reflexive.</td>
<td>$R$ is an equivalence relation.</td>
</tr>
<tr>
<td>• $\forall x \in A$. $xRx$</td>
<td>$R$ is reflexive.</td>
</tr>
<tr>
<td>• $R$ is cyclic.</td>
<td>$R$ is symmetric.</td>
</tr>
<tr>
<td>• $xRy \land yRz \rightarrow zRx$</td>
<td>$R$ is transitive.</td>
</tr>
</tbody>
</table>
Lemma 2: If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ is reflexive.</td>
<td>$R$ is transitive.</td>
</tr>
<tr>
<td>$\forall x \in A, xRx$</td>
<td>If $arb$ and $brc$, then $arc$.</td>
</tr>
<tr>
<td>$R$ is cyclic.</td>
<td></td>
</tr>
<tr>
<td>$xRy \land yRz \rightarrow zRx$</td>
<td></td>
</tr>
</tbody>
</table>
Lemma 2: If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

What We’re Assuming

$R$ is reflexive.

$\forall x \in A. \ xRx$

$R$ is cyclic.

$xRy \land yRz \rightarrow zRx$

What We Need To Show

• $R$ is transitive.

• If $arb$ and $brc$, then $arc$.

\[ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\rightarrow \\
\text{b}
\end{array} \\
\begin{array}{c}
\text{c}
\end{array}
\end{array} \]
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

**What We’re Assuming**

- $R$ is reflexive.
- $\forall x \in A. \ xRx$
- $R$ is cyclic.
- $xRy \land yRz \rightarrow zRx$

**What We Need To Show**

- $R$ is transitive.
- If $arb$ and $brc$, then $arc$.  

![Diagram showing cyclic relation with nodes a, b, and c, illustrating the cyclic property with arrows a -> b, b -> c, and c -> a.]
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

**What We’re Assuming**

- $R$ is reflexive.
  - $\forall x \in A. \ xRx$
- $R$ is cyclic.
  - $xRy \land yRz \rightarrow zRx$
- $R$ is symmetric
  - $xRy \rightarrow yRx$

**What We Need To Show**

- $R$ is transitive.
  - If $arb$ and $brc$, then $arc$. 

```
  a    b
   ↓   ↓
  b    c
   ↓   ↓
  c    a
```
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b, c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required. ■
Lemma 2: If \( R \) is a binary relation over a set \( A \) that is cyclic and reflexive, then \( R \) is an equivalence relation.

Proof:
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric.
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds.
Lemma 2: If \( R \) is a binary relation over a set \( A \) that is cyclic and reflexive, then \( R \) is an equivalence relation.

Proof: Let \( R \) be an arbitrary binary relation over a set \( A \) that is cyclic and reflexive. We need to prove that \( R \) is an equivalence relation. To do so, we need to show that \( R \) is reflexive, symmetric, and transitive. Since we already know by assumption that \( R \) is reflexive, we just need to show that \( R \) is symmetric and transitive.

First, we'll prove that \( R \) is symmetric. To do so, pick any arbitrary \( a, b \in A \) where \( aRb \) holds. We need to prove that \( bRa \) is true.
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds.
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive.
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b,$ and $c$ be any elements of $A$ where $aRb$ and $bRc$. 
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b,$ and $c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. 
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b, \text{ and } c$ be any elements of $A$ where $aRb \text{ and } bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb \text{ and } bRc$ we see that $cRa$. 
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b, c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required.
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

**Proof:** Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b,$ and $c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required. ■
Lemma 2:

If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof:

Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b, \text{ and } c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $a Rc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required. ■

Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b,$ and $c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required. ■
Lemma 2:

If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to show that $R$ is an equivalence relation. To do so, we need to show that it is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b,$ and $c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required. ■

Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That's normal – it's actually quite rare to see first-order logic in written proofs.
Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

First, we'll prove that $R$ is symmetric. To do so, pick any arbitrary $a, b \in A$ where $aRb$ holds. We need to prove that $bRa$ is true. Since $R$ is reflexive, we know that $aRa$ holds. Therefore, by cyclicity, since $aRa$ and $aRb$, we learn that $bRa$, as required.

Next, we'll prove that $R$ is transitive. Let $a, b, \text{ and } c$ be any elements of $A$ where $aRb$ and $bRc$. We need to prove that $aRc$. Since $R$ is cyclic, from $aRb$ and $bRc$ we see that $cRa$. Earlier, we showed that $R$ is symmetric. Therefore, from $cRa$ we see that $aRc$ is true, as required. ■
Refining Your Proofwriting

- When writing proofs about terms with formal definitions, you must call back to those definitions.
  - Use the first-order definition to see what you’ll assume and what you’ll need to prove.
- When writing proofs about terms with formal definitions, you should not include any first-order logic in your proofs.
  - Although you won’t use any FOL notation in your proofs, your proof implicitly calls back to the FOL definitions.
- You’ll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!
Time-Out for Announcements!
My Office Hours

• Oops! I forgot to put my office hours into the OH timetable.
• They’re Thursdays, 2:15PM – 4:15PM, in Gates 167.
• Feel free to stop on by!
Problem Set One Graded

- We’ve finished grading Problem Set One. Feedback is available on GradeScope.

- Here’s the distribution:
What To Do Next

- **Review the grader’s feedback.** We try to leave detailed feedback on each problem. Look over our notes and see if you can find some concrete, tangible ways to improve going forward.

- **Don’t get discouraged.** This problem set is downweighted relative to the other problem sets this quarter. For the overwhelming majority of you, this is your first time writing proofs. Don’t extrapolate from just one data point – figure out where to focus your efforts, and try to make new mistakes each time.
THE ROAD TO WISDOM

The road to wisdom?—Well, it's plain and simple to express:
Err
and err
and err again,
but less
and less
and less.

— Piet Hein

CS legend Don Knuth has this poem on the wall of his house.

And this guy is interesting. You should look him up.
Problem Set Two

- Problem Set Two is due on Friday at the start of class.
  - Have questions? Stop by office hours or ask on Piazza!
- We’ve released a handout containing a first-order logic translation checklist. We highly recommend reviewing your translations using that checklist before submitting!
Your Questions
“Why is biocomp rare? Will it be big in the future?”

Biocomputation is a really big field right now! The number of CS majors graduating with the biocomputation track has exploded over the past few years (much faster than the general CS major), which is really exciting!

I’ve seen some amazing talks by Gill Bejerano and Serafim Batzoglu about the work they’re doing working out how genes control one another and how to sequence cancer genomes, and it really feels like science fiction. This is a very cool area to explore!
“Do you think Silicon Valley has a good moral compass?”

I think it’s useful to think about things like this from a few perspectives. First, what does the leadership at a company value? Second, what are the incentive structures? Third, what are the broader values of the community?

There are many areas where I think the tech industry has things right. There are many areas where I think the tech industry has things wrong. But I wouldn’t necessarily attribute it to a “moral compass.” I (personally) think aggregate behavior is best explained by the three above factors.
Back to CS103!
Prerequisite Structures
The CS Core

Systems

CS106B
Programming Abstractions

CS107
Computer Organization and Systems

CS110
Principles of Computer Systems

Theory

CS103
Mathematical Foundations of Computing

CS109
Intro to Probability for Computer Scientists

CS161
Design and Analysis of Algorithms
Pancakes

Everyone's got a pancake recipe. This one comes from Food Wishes (http://foodwishes.blogspot.com/2011/08/grandma-kellys-good-old-fashioned.html).

Ingredients

- 1 1/2 cups all-purpose flour
- 3 1/2 tsp baking powder
- 1 tsp salt
- 1 tbsp sugar
- 1 1/4 cup milk
- 1 egg
- 3 tbsp butter, melted

Directions

1. Sift the dry ingredients together.
2. Stir in the butter, egg, and milk. Whisk together to form the batter.
3. Heat a large pan or griddle on medium-high heat. Add some oil.
4. Make pancakes one at a time using 1/4 cup batter each. They're ready to flip when the centers of the pancakes start to bubble.
Measure Flour
Measure Sugar
Measure Baking Pwdr
Measure Salt

Heat Griddle
Beat Egg
Combine Dry Ingredients
Melt Butter
Measure Milk

Oil Griddle
Add Wet Ingredients

Make Pancakes

Serve Pancakes
Relations and Prerequisites

Let's imagine that we have a prerequisite structure with no circular dependencies.

We can think about a binary relation $R$ where $aRb$ means

"$a$ must happen before $b$"

What properties of $R$ could we deduce just from this?
\( aRa \)

\( aRb \land bRc \rightarrow aRc \)

\( aRb \rightarrow bRa \)
∀a ∈ A. aR\not\sim a

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)

∀a ∈ A. ∀b ∈ A. (aRb → b\not\sim a)
∀a ∈ A. aRa

Transitivity

∀a ∈ A. ∀b ∈ A. (aRb → bRa)
\[ \forall a \in A. aRa \]

Transitivity

\[ \forall a \in A. \forall b \in A. (aRb \rightarrow bRa) \]
Irreflexivity

- Some relations *never* hold from any element to itself.
- As an example, \( x \not\prec x \) for any \( x \).
- Relations of this sort are called *irreflexive*.
- Formally speaking, a binary relation \( R \) over a set \( A \) is irreflexive if the following first-order logic statement is true about \( R \):

\[
\forall a \in A. \ a \not\mathrel{R} a
\]

("No element is related to itself.")
Irreflexivity Visualized

∀a ∈ A. a赧a

(“No element is related to itself.”)
Is this relation reflexive?
$\forall a \in A. \ aRa$

("Every element is related to itself.")
\[ \forall a \in A. \ aRa \]

(“Every element is related to itself.”)
∀a ∈ A. aRa

("Every element is related to itself.")
Is this relation irreflexive?
\( \forall a \in A. \ a \not R a \)

(“No element is related to itself.”)
$\forall a \in A. \ a \not\mathrel{R} a$

(“No element is related to itself.”)
∀a ∈ A. aR̸a
("No element is related to itself.")
Reflexivity and Irreflexivity

• Reflexivity and irreflexivity are not opposites!
• Here's the definition of reflexivity:
  \[ \forall a \in A. \ aRa \]

• What is the negation of the above statement?
  \[ \exists a \in A. \ aRa \]

• What is the definition of irreflexivity?
  \[ \forall a \in A. \ a \not\in Ra \]
∀a ∈ A. aRa

Transitivity

∀a ∈ A. ∀b ∈ A. (aRb → b∀a)
Irreflexivity

∀a ∈ A. ∀b ∈ A. (aRb → b¬Ra)

Transitivity
Irreflexivity

\[ \forall a \in A. \forall b \in A. (aRb \rightarrow b\not\mathrel{R}a) \]

Transitivity
Asymmetry

• In some relations, the relative order of the objects can never be reversed.
• As an example, if \( x < y \), then \( y \not< x \).
• These relations are called \textit{asymmetric}.
• Formally: a binary relation \( R \) over a set \( A \) is called \textit{asymmetric} if the following first-order logic statement is true about \( R \):

\[
\forall a \in A. \forall b \in A. (aRb \rightarrow b \not R a)
\]

(“If \( a \) relates to \( b \), then \( b \) does not relate to \( a \).”)
∀a ∈ A. ∀b ∈ A. (aRb → bR̸a)
("If a relates to b, then b does not relate to a.")
Question to Ponder: Are symmetry and asymmetry opposites of one another?
Irreflexivity

∀a ∈ A. ∀b ∈ A. (aRb → b\not{Ra})

Transitivitiy
Irreflexivity

Transitivity

Asymmetry
Strict Orders

• A strict order is a relation that is irreflexive, asymmetric and transitive.

• Some examples:
  • $x < y$.
  • $a$ can run faster than $b$.
  • $A \subset B$ (that is, $A \subseteq B$ and $A \neq B$).

• Strict orders are useful for representing prerequisite structures and have applications in complexity theory (measuring notions of relative hardness) and algorithms (searching and sorting).
Strict Order Proofs

- Let's suppose that you're asked to prove that a binary relation is a strict order.
- Calling back to the definition, you could prove that the relation is asymmetric, irreflexive, and transitive.
- However, there's a slightly easier approach we can use instead.
Theorem: Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.
**Theorem:** Let \( R \) be a binary relation over a set \( A \). If \( R \) is asymmetric, then \( R \) is irreflexive.

What's the high-level structure of this proof?
**Theorem**: Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

What’s the high-level structure of this proof?

∀$R$. (Asymmetric($R$) $\rightarrow$ Irreflexive($R$))
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

What's the high-level structure of this proof?

\[ \forall R. \ (\text{Asymmetric}(R) \rightarrow \text{Irreflexive}(R)) \]

Therefore, we’ll choose an arbitrary asymmetric relation $R$, then go and prove that $R$ is irreflexive.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

What's the high-level structure of this proof?

∀$R$. (Asymmetric($R$) → Irreflexive($R$))

Therefore, we'll choose an arbitrary asymmetric relation $R$, then go and prove that $R$ is irreflexive.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.
Theorem: Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

Proof: Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. Because the relation $R$ is asymmetric and $xRx$ holds, we conclude that $xRx$ holds as well. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be reflexive. ■

What is the definition of irreflexivity?

∀$x \in A$.

What is the negation of this statement?

∃$x \in A$.

So let's suppose that there is some element $x \in A$ such that $xRx$ and proceed from there.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive. To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. Because the relation $R$ is asymmetric and $xRx$ holds, we conclude that $xRx$ holds as well. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive. ■

**What is the definition of irreflexivity?**

$$\forall x \in A. \; x \not\in R x$$

**What is the negation of this statement?**

$$\exists x \in A. \; x \in R x$$

So let's suppose that there is some element $x \in A$ such that $xRx$ and proceed from there.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.

Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. Because the relation $R$ is asymmetric and $xRx$ holds, we conclude that $xRx$ holds as well. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive. ■
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. Because the relation $R$ is asymmetric and $xRx$ holds, we conclude that $x \not R x$ holds as well. However, this is impossible, since we can't have both $xRx$ and $x \not R x$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive. ■

What is the definition of irreflexivity?

$$\forall x \in A. x \not R x$$

What is the negation of this statement?

$$\exists x \in A. x R x$$
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. Because the relation $R$ is asymmetric and $xRx$ holds, we conclude that $xRx$ holds as well. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be reflexive. ■

What is the definition of irreflexivity?

\[ \forall x \in A. \, x \not\!Rx \]

What is the negation of this statement?

\[ \exists x \in A. \, xRx \]

So let's suppose that there is some element $x \in A$ such that $xRx$ and proceed from there.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive. ■
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. 
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds. We know by assumption that $xRx$ is true, so we conclude that $xRx$ holds.
**Theorem:** Let \( R \) be a binary relation over a set \( A \). If \( R \) is asymmetric, then \( R \) is irreflexive.

**Proof:** Let \( R \) be an arbitrary asymmetric binary relation over a set \( A \). We will prove that \( R \) is irreflexive.

To do so, we will proceed by contradiction. Suppose that \( R \) is not irreflexive. That means that there must be some \( x \in A \) such that \( xRx \).

Since \( R \) is asymmetric, we know for any \( a, b \in A \) that if \( aRb \) holds, then \( bRa \) holds. Plugging in \( a=x \) and \( b=x \), we see that if \( xRx \) holds, then \( xRx \) holds. We know by assumption that \( xRx \) is true, so we conclude that \( xRx \) holds. However, this is impossible, since we can't have both \( xRx \) and \( xRx \).
Theorem: Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

Proof: Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds. We know by assumption that $xRx$ is true, so we conclude that $xRx$ holds. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds. We know by assumption that $xRx$ is true, so we conclude that $xRx$ holds. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive.
**Theorem:** Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

**Proof:** Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds. We know by assumption that $xRx$ is true, so we conclude that $xRx$ holds. However, this is impossible, since we can't have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive. ■
**Theorem:** If a binary relation $R$ is asymmetric and transitive, then $R$ is a strict order.

**Proof:** Let $R$ be a binary relation that is asymmetric and transitive. Since $R$ is asymmetric, by our previous theorem we know that $R$ is also irreflexive. Therefore, $R$ is asymmetric, irreflexive, and transitive, so by definition $R$ is a strict order. $\blacksquare$

To prove that some binary relation $R$ is a strict order, you can just prove that $R$ is asymmetric and transitive. In the next problem set, you'll see an even simpler technique!
Drawing Strict Orders
<table>
<thead>
<tr>
<th>Gold</th>
<th>Silver</th>
<th>Bronze</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>37</td>
<td>38</td>
</tr>
<tr>
<td>27</td>
<td>23</td>
<td>17</td>
</tr>
<tr>
<td>26</td>
<td>18</td>
<td>26</td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td>17</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>21</td>
</tr>
<tr>
<td>10</td>
<td>18</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
(g₁, s₁, b₁) R (g₂, s₂, b₂) \text{ if } g₁ < g₂ \land s₁ < s₂ \land b₁ < b₂
\((g_1, s_1, b_1) \text{ } R \text{ } (g_2, s_2, b_2)\) \quad \text{if} \quad g_1 < g_2 \text{ } \land \text{ } s_1 < s_2 \text{ } \land \text{ } b_1 < b_2
\[(g_1, s_1, b_1) \, R \, (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2\]
\((g_1, s_1, b_1) \mathcal{R} (g_2, s_2, b_2)\) if \(g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2\)
More Medals

\[(g_1, s_1, b_1) \text{ } R \text{ } (g_2, s_2, b_2) \text{ if } g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2\]

Fewer Medals
More Medals

(46, 37, 38)

(26, 18, 26)  (19, 18, 19)  (27, 23, 17)

(12, 8, 21)  (17, 10, 15)  (10, 18, 14)

Fewer Medals

\((g_1, s_1, b_1) \, R \, (g_2, s_2, b_2)\) \quad \text{if} \quad g_1 < g_2 \, \land \, s_1 < s_2 \, \land \, b_1 < b_2
\[(g_1, s_1, b_1) \, R \, (g_2, s_2, b_2) \text{ if } g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2\]
Hasse Diagrams

- A **Hasse diagram** is a graphical representation of a strict order.
- Elements are drawn from bottom-to-top.
- Higher elements are bigger than lower elements: by **asymmetry**, the edges can only go in one direction.
- No redundant edges: by **transitivity**, we can infer the missing edges.
\[(g_1, s_1, b_1) T (g_2, s_2, b_2) \]

if

\[5g_1 + 3s_1 + b_1 < 5g_2 + 3s_2 + b_2\]
\[(g_1, s_1, b_1) \quad T \quad (g_2, s_2, b_2)\]

if

\[5g_1 + 3s_1 + b_1 < 5g_2 + 3s_2 + b_2\]
\((g_1, s_1, b_1) \leq (g_2, s_2, b_2)\) if
\[5g_1 + 3s_1 + b_1 < 5g_2 + 3s_2 + b_2\]
\[(g_1, s_1, b_1) \cup (g_2, s_2, b_2) \quad \text{if} \quad g_1 + s_1 + b_1 < g_2 + s_2 + b_2\]
\((g_1, s_1, b_1) \cup (g_2, s_2, b_2)\)

if

\[g_1 + s_1 + b_1 < g_2 + s_2 + b_2\]
(g₁, s₁, b₁) \cup (g₂, s₂, b₂)

\text{if}

g₁ + s₁ + b₁ < g₂ + s₂ + b₂
Hasse Artichokes

\[ x \succ y \text{ if } x \text{ must be eaten before } y \]
Hasse Artichokes

\[ x \text{R} y \text{ if } x \text{ must be eaten before } y \]
The Meta Strict Order

- Irreflexivity
- Asymmetry
- Transitivity
- Reflexivity
- Symmetry

\( aRb \) if \( a \) is less specific than \( b \)
The Binary Relation Editor