Binary Relations

Part II

Outline for Today

- ***Finish from Last Time***
  - Pt. 3 of our proof that ~ is an equivalence relation
- ***Properties of Equivalence Relations***
  - What's so special about those three rules?
- ***Strict Orders***
  - A different type of mathematical structure
- ***Hasse Diagrams***
  - How to visualize rankings

Finish from Last Time

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even. Therefore, $a \sim a$ holds, as required. ■

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

New Stuff!

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation ~ is transitive.

Which of the following works best as the introduction to this proof?

A. Pick an arbitrary a and b from A where $a \sim b$. We'll prove $b \sim a$.
B. Consider any $a, b, c \in A$ where $a \sim b$, $b \sim c$, and $a \sim c$.
C. Choose an $a, b, c \in A$. We will prove $a \sim b$, $b \sim c$, and $a \sim c$.
D. Take any $a, b, c \in A$ where $a \sim b$ and $b \sim c$; we'll prove $a \sim c$.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, or **D**.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation ~ is transitive.

Proof:

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

Therefore, we'll choose arbitrary integers a, b, and c where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required. ■

First-Order Logic and Proofs

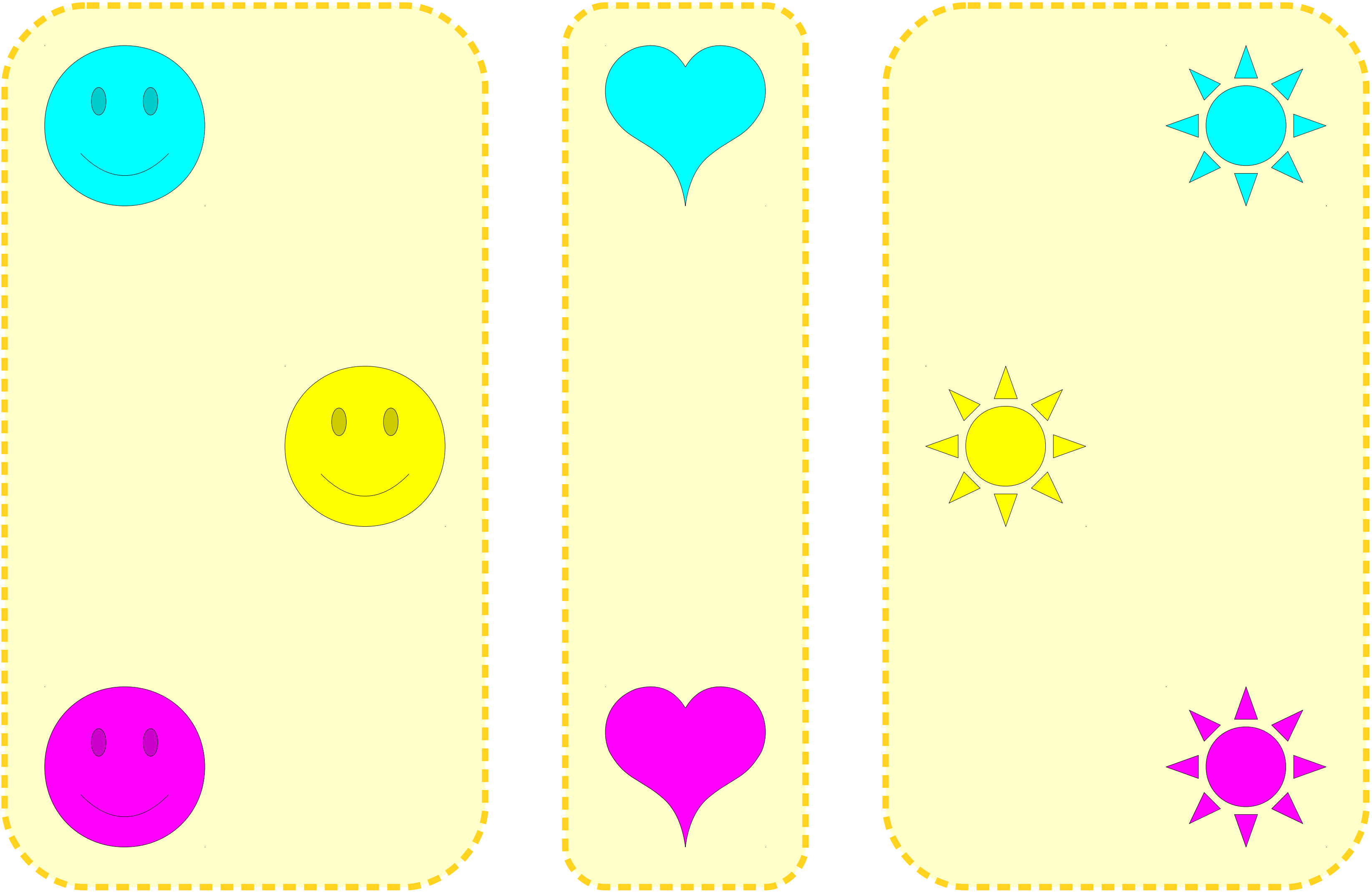
- First-order logic is an excellent tool for giving formal definitions to key terms.
- While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.
- Follow the example of these proofs:
  - Use the FOL definitions to determine what to assume and what to prove.
  - Write the proof in plain English using the conventions we set up in the first week of the class.
- ***Please, please, please, please, please internalize the contents of this slide!***

$$\forall a \in A.\ aRa$$

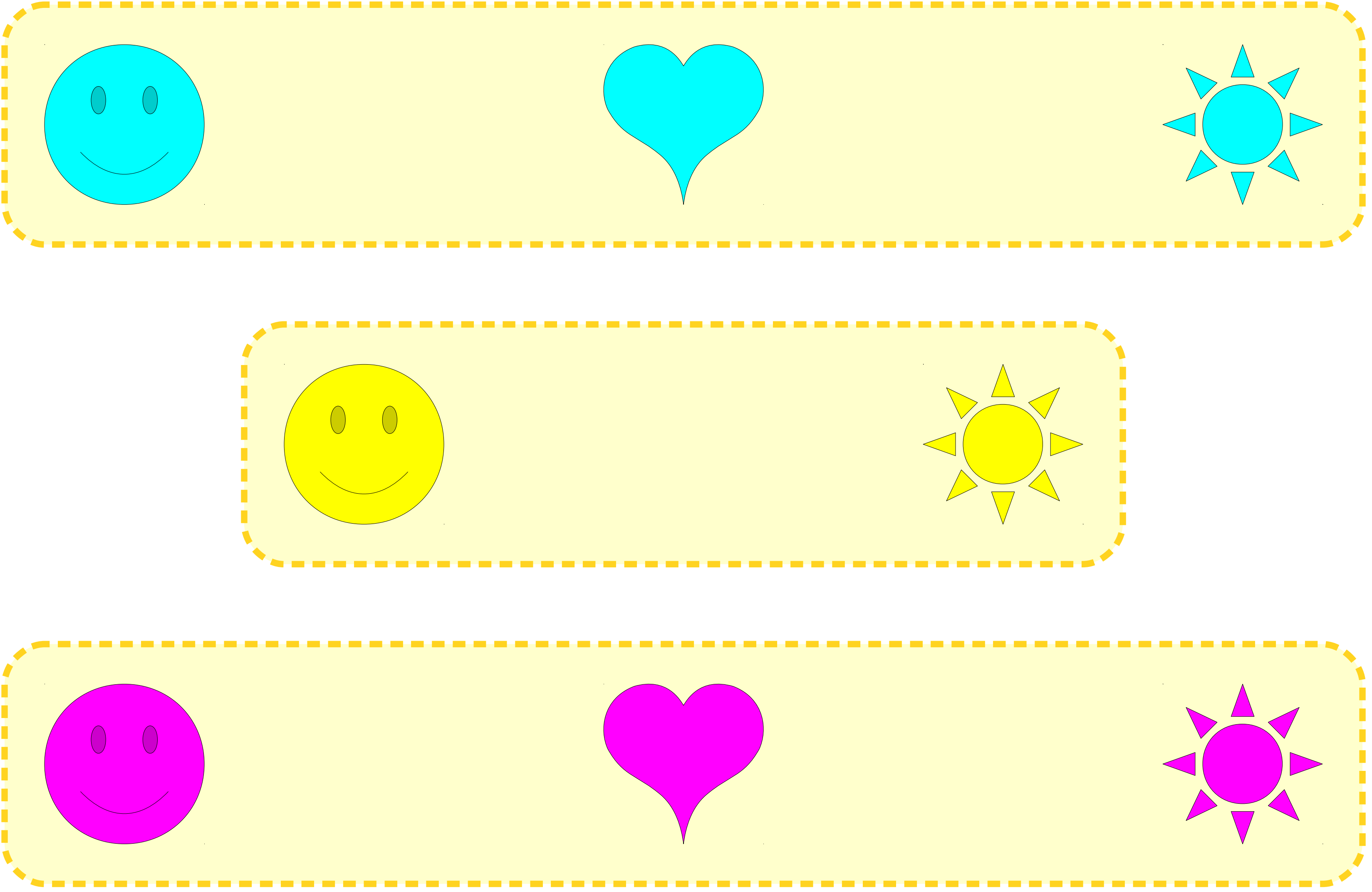
$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

Properties of Equivalence Relations



xRy if x and y have the same shape



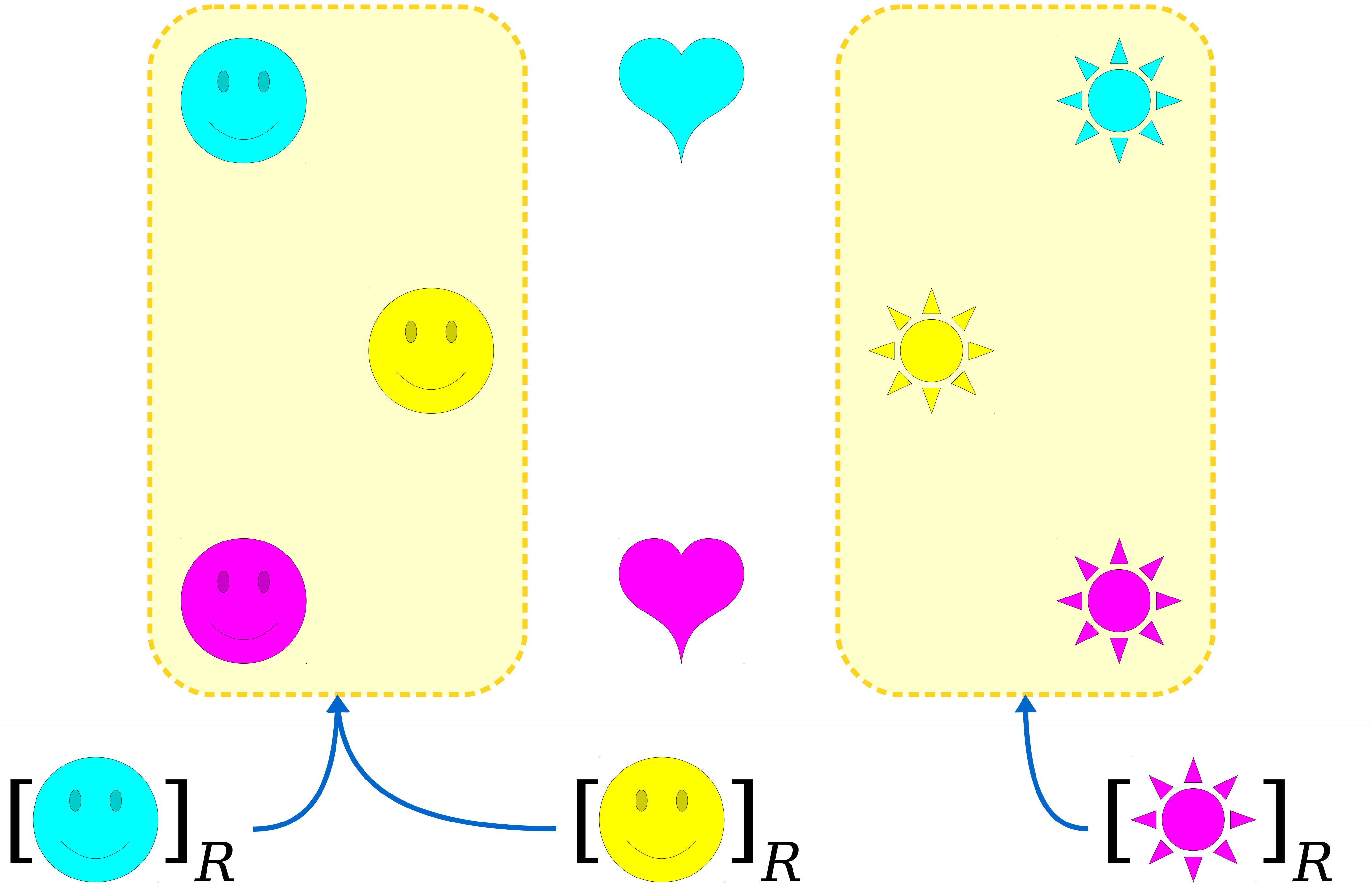
xTy if x and y have the same color

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{ \, y \in A \mid xRy \, \}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R.



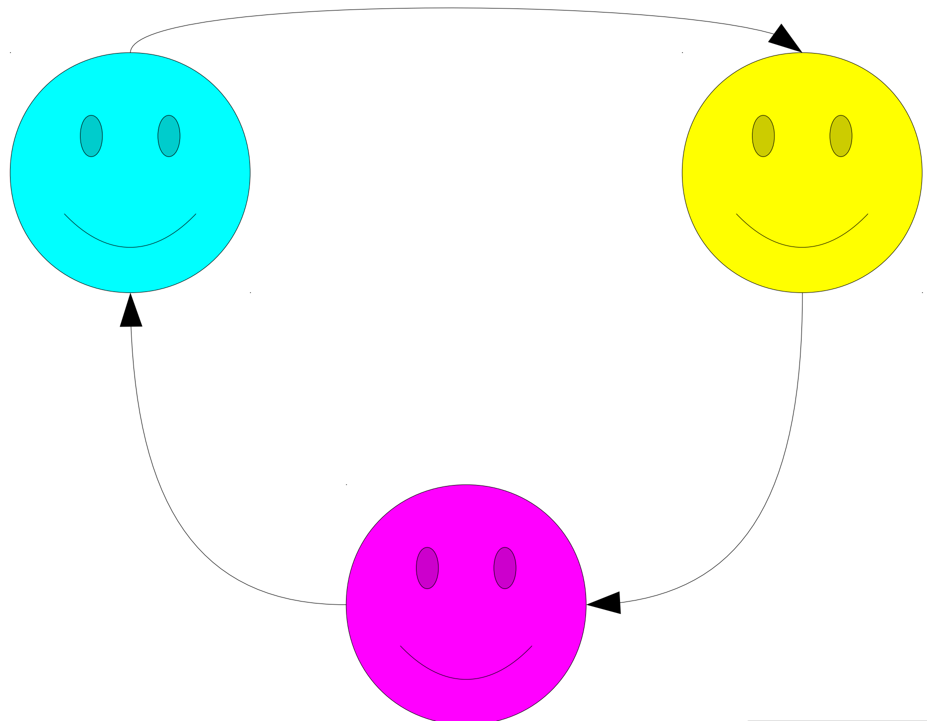
xRy if x and y have the same shape

The Fundamental Theorem of Equivalence Relations: Let R be an equivalence relation over a set A. Then every element $a \in A$ belongs to exactly one equivalence class of R.

How'd We Get Here?

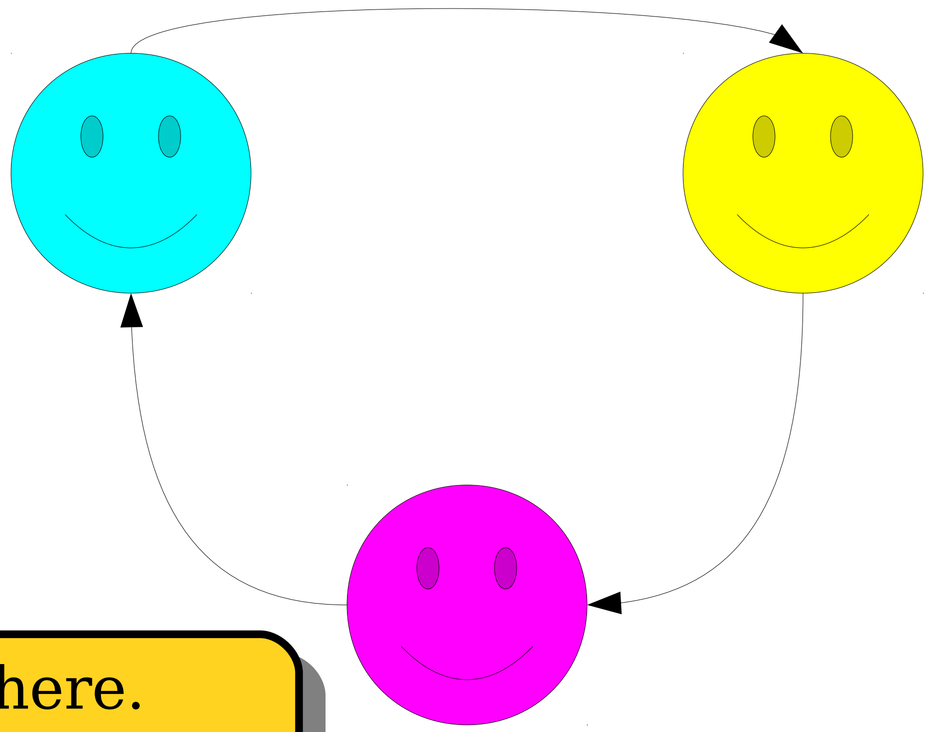
- We discovered equivalence relations by thinking about ***partitions*** of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- ***Question:*** What's so special about these three rules?

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$



A binary relation with this property is called ***cyclic***.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow cRa)$$



Let R be the relation depicted here.
How many of the following claims are true?

R is reflexive.
R is symmetric.
R is transitive.
R is an equivalence relation.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

Theorem: A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming	*What We Need To Show*

• R is an equivalence relation.
  • R is reflexive.
  • R is symmetric.
  • R is transitive.
	• R is reflexive.
	• R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

R is reflexive.

R is cyclic.

- If aRb and bRc, then cRa.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

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R is an equivalence relation.

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R is symmetric.

- R is transitive.

What We Need To Show

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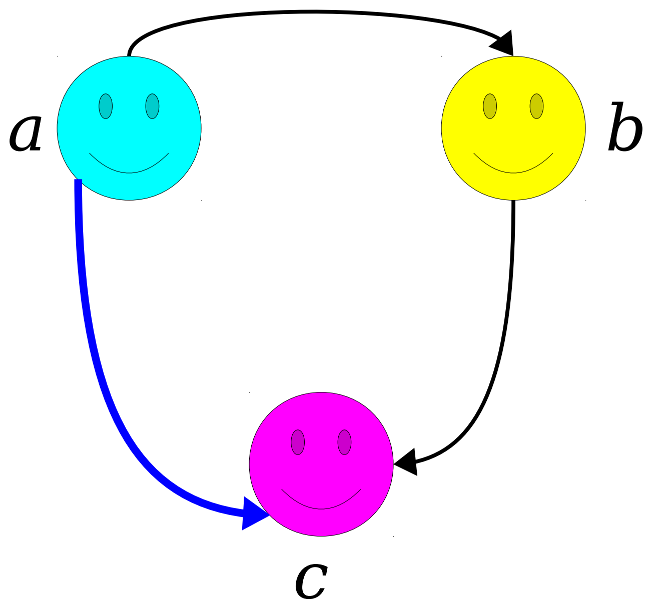

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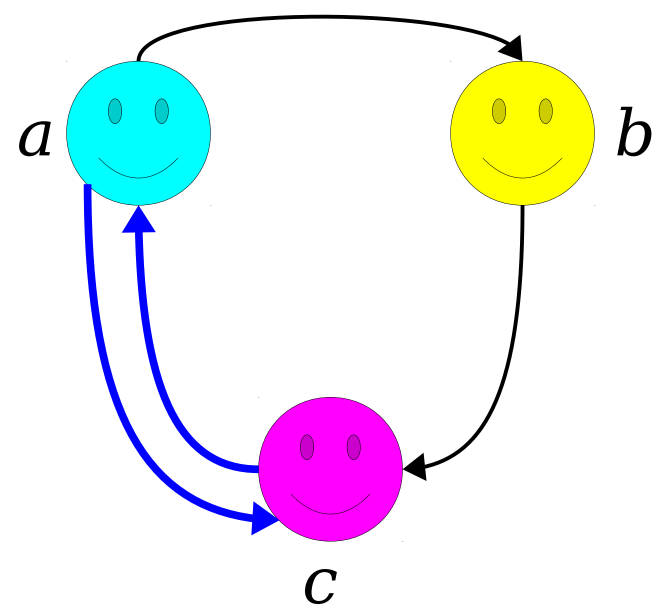
- R is symmetric.

R is transitive.



Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

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Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is
reflexive, symmetr
already know that
that R is cyclic.

Notice how the first few sentences of this proof mirror the structure of what needs to be proved. We're just following the templates from the first week of class!

To prove that R is ... A
where aRb and bR
Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Lemma ... en R
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Proof: L... me
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Since ...
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that R is cyclic.

Notice how this setup mirrors the first-order definition of cyclicity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That's normal – it's actually quite rare to see first-order logic in written proofs.

Lem … en R
is …

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

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Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
- R is cyclic.

What We Need To Show

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming	*What We Need To Show*
• R is reflexive. • R is cyclic.	R is an equivalence relation. R is reflexive. • R is symmetric. R is transitive.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

R is reflexive.

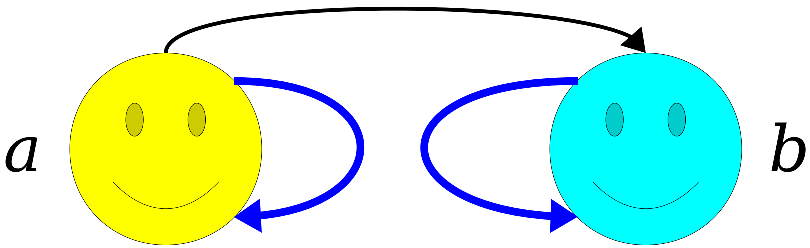
- $\forall x \in A.\ xRx$

R is cyclic.

$xRy \land yRz \to zRx$

What We Need To Show

- R is symmetric.
  - If aRb, then bRa.



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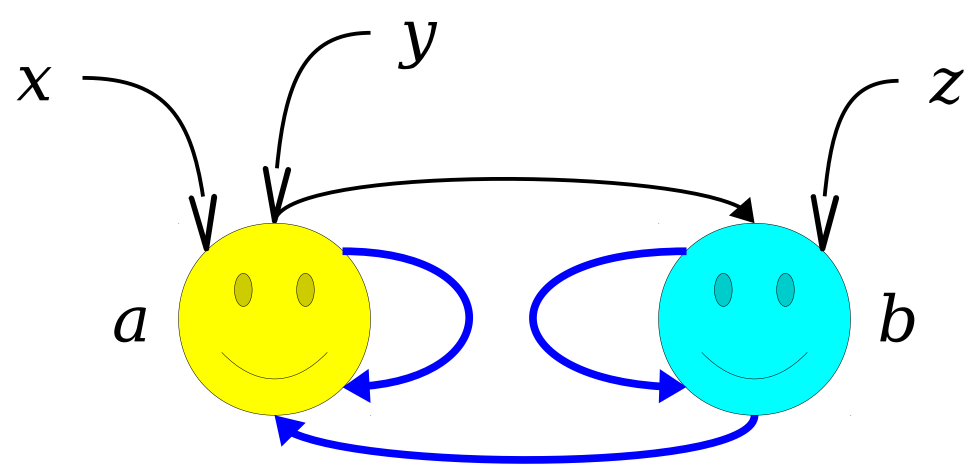

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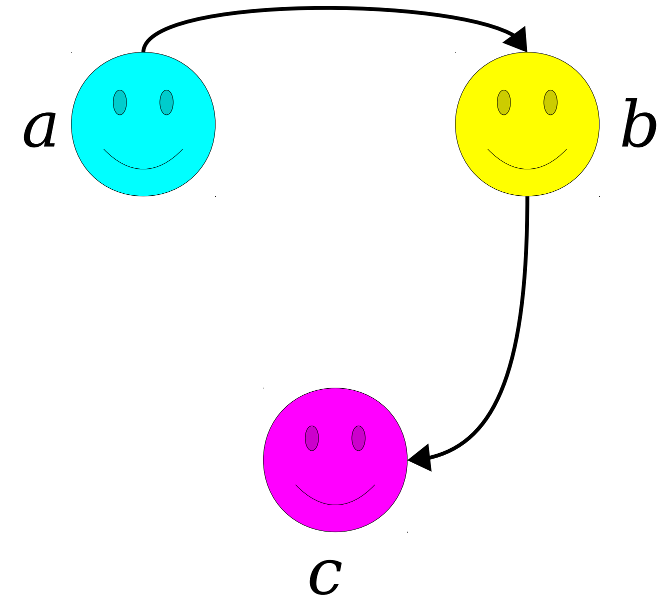
- R is reflexive.
  - $\forall x \in A.\ xRx$
- R is cyclic.
  - $xRy \land yRz \to zRx$

What We Need To Show

- R is transitive.
  - If aRb and bRc, then aRc.



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$xRy \land yRz \to zRx$

What We Need To Show

- R is transitive.
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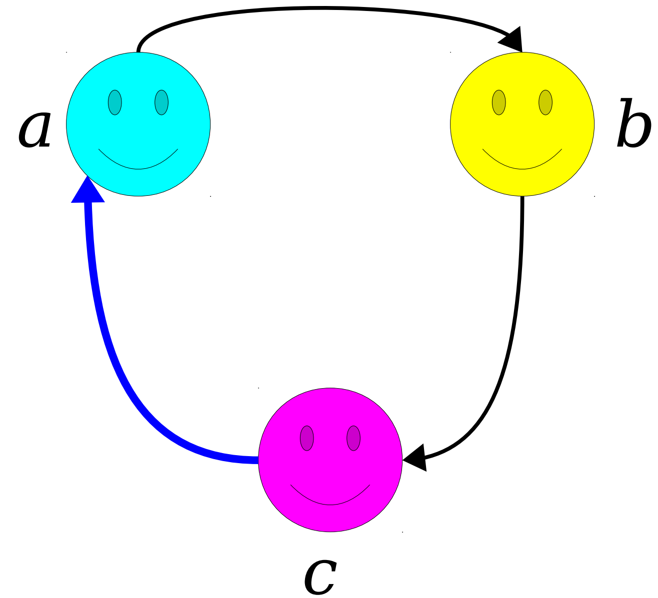


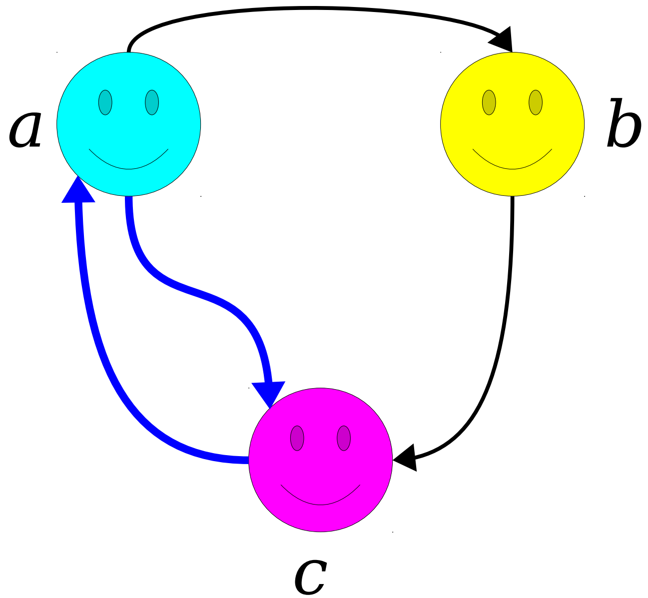
Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

What We're Assuming

- R is reflexive.
  - ∀x ∈ A. xRx
- R is cyclic.
  - xRy ∧ yRz → zRx
- R is symmetric
  - xRy → yRx

What We Need To Show

- R is transitive.
  - If aRb and bRc, then aRc.



Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

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Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

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Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equi… is refle… know… show…

Notice how this setup mirrors the first-order definition of transitivity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow aRc)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

First … y arbit … at bRa … s. Then … t bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

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Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

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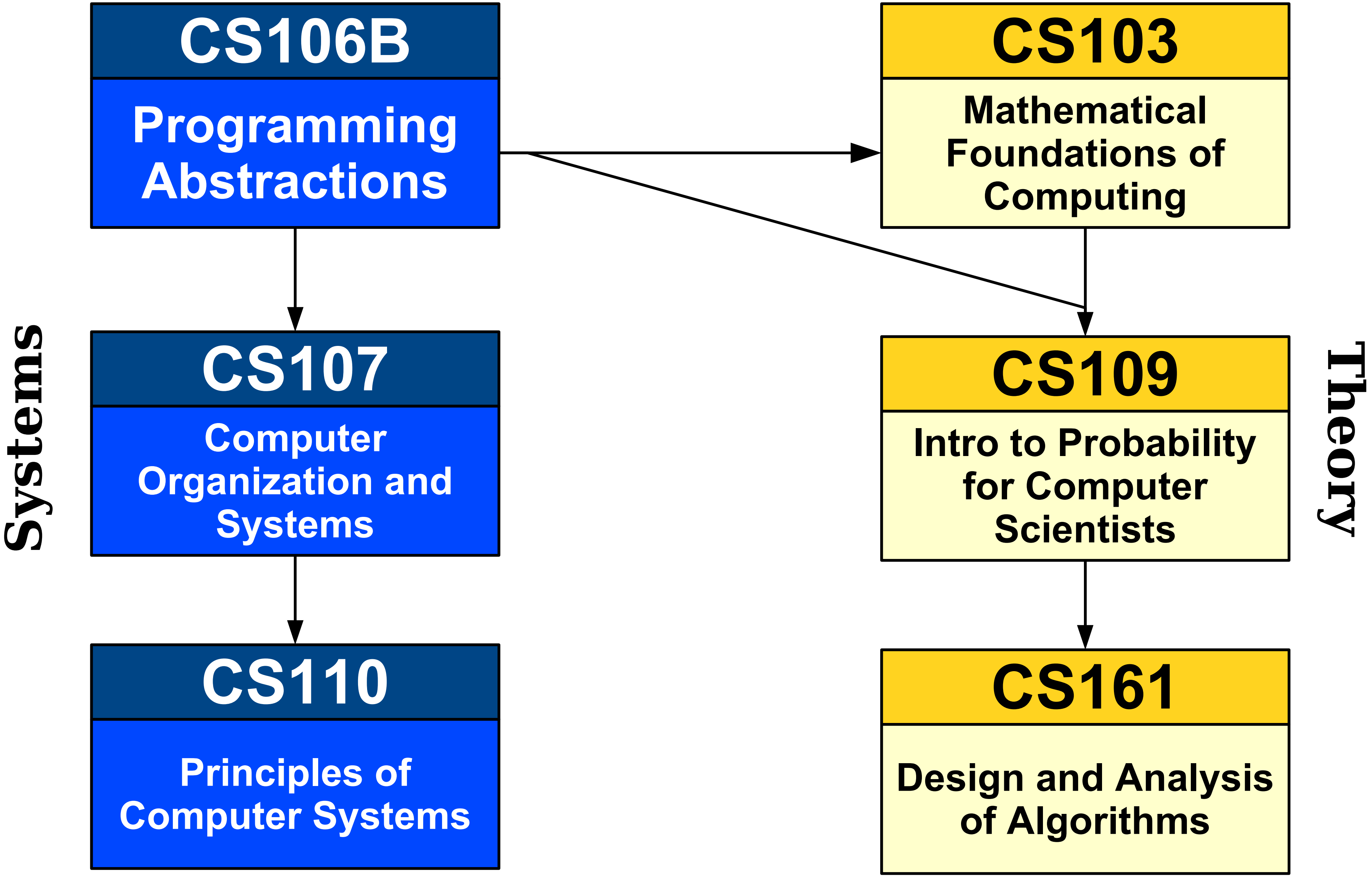
Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Refining Your Proofwriting

- When writing proofs about terms with formal definitions, you ***must*** call back to those definitions.
  - Use the first-order definition to see what you'll assume and what you'll need to prove.
- When writing proofs about terms with formal definitions, you ***must not*** include any first-order logic in your proofs.
  - Although you won't use any FOL *notation* in your proofs, your proof implicitly calls back to the FOL definitions.
- You'll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!

Prerequisite Structures

The CS Core


CS106B
Programming Abstractions
CS103
Mathematical Foundations of Computing
Systems
CS107
Computer Organization and Systems
CS109
Intro to Probability for Computer Scientists
Theory
CS110
Principles of Computer Systems
CS161
Design and Analysis of Algorithms

Pancakes

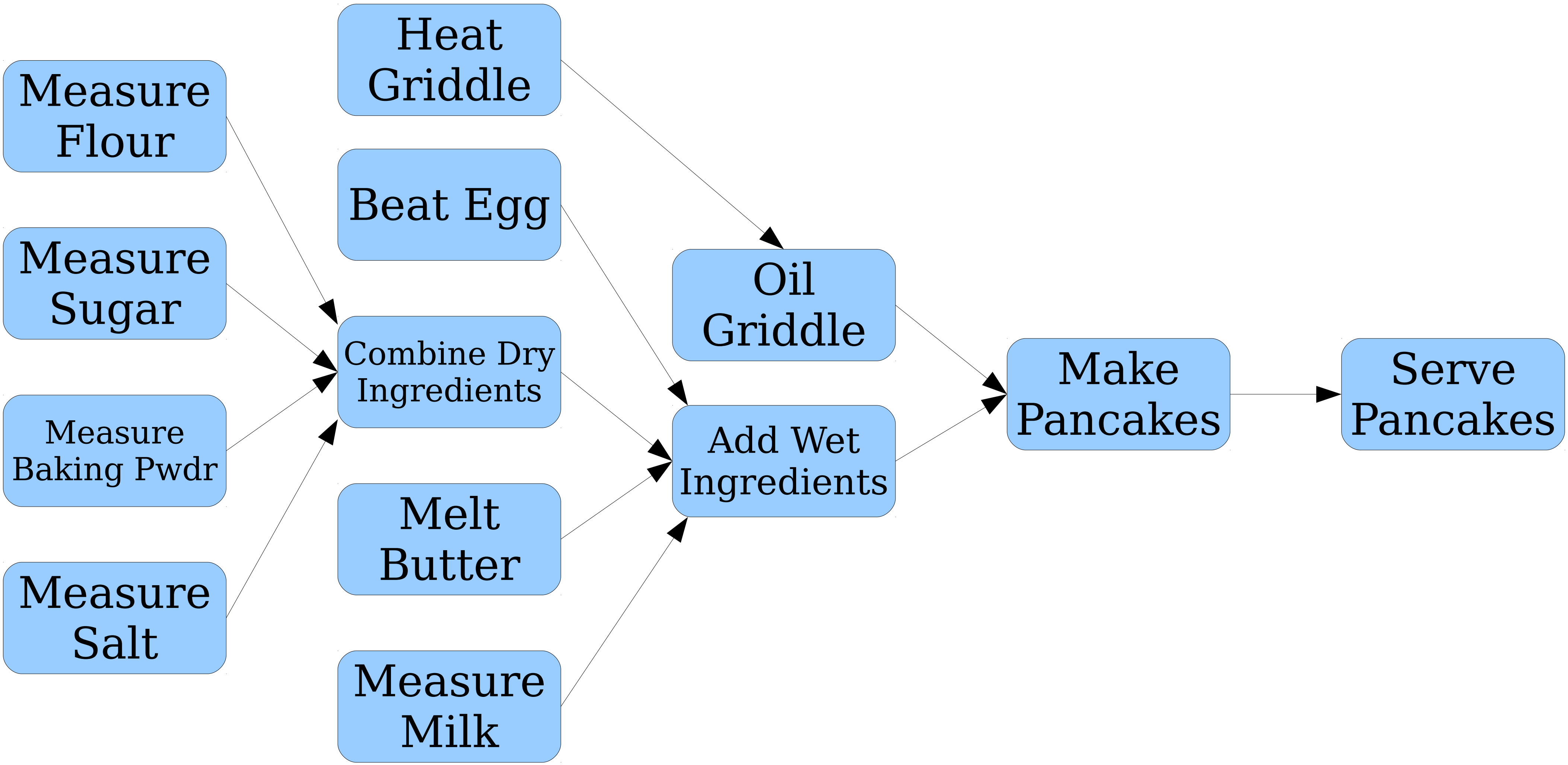
Everyone's got a pancake recipe. This one comes from Food Wishes (http://foodwishes.blogspot.com/2011/08/grandma-kellys-good-old-fashioned.html).

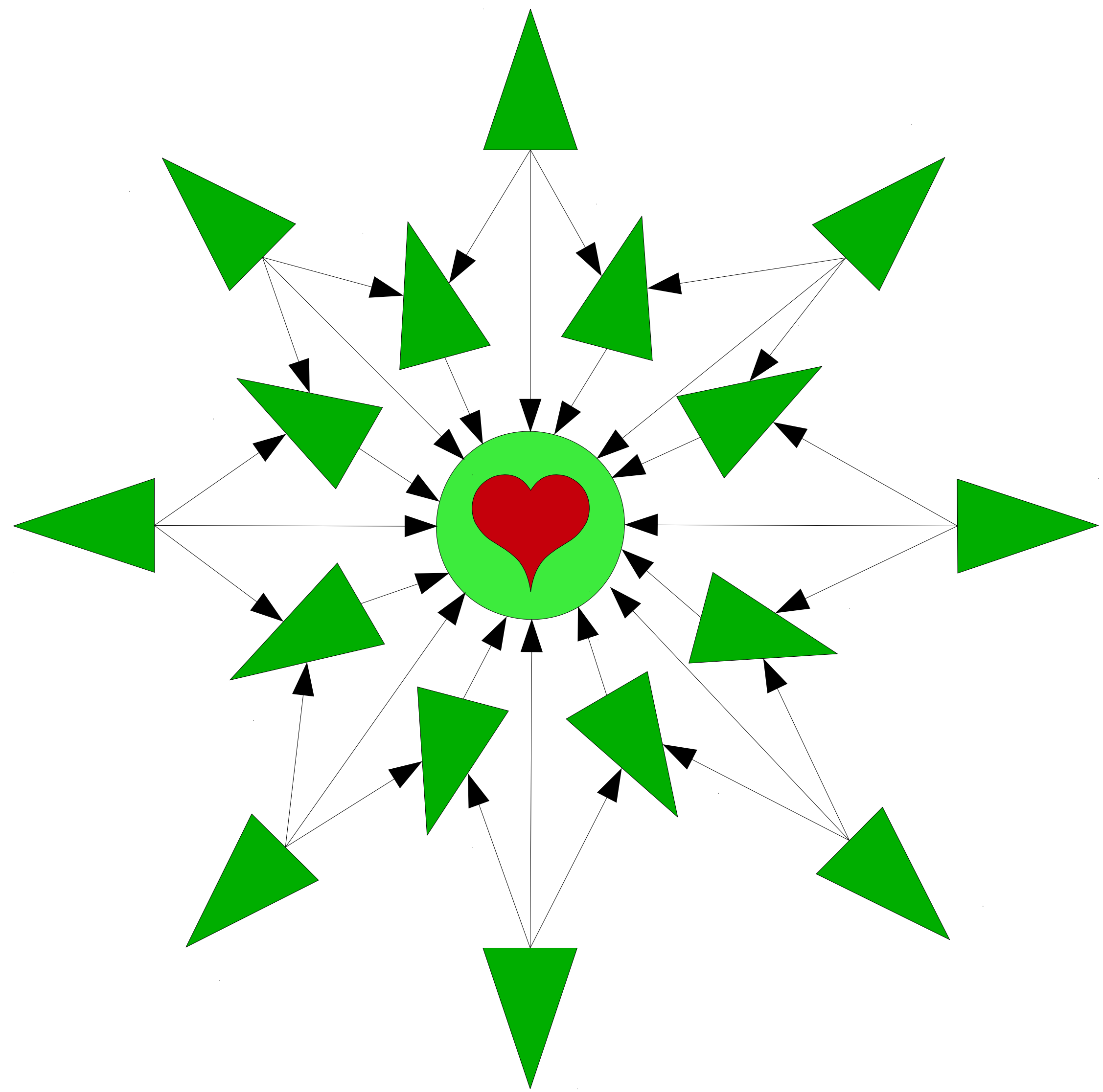
Ingredients

- 1 1/2 cups all-purpose flour
- 3 1/2 tsp baking powder
- 1 tsp salt
- 1 tbsp sugar
- 1 1/4 cup milk
- 1 egg
- 3 tbsp butter, melted

Directions

1. Sift the dry ingredients together.
2. Stir in the butter, egg, and milk. Whisk together to form the batter.
3. Heat a large pan or griddle on medium-high heat. Add some oil.
4. Make pancakes one at a time using 1/4 cup batter each. They're ready to flip when the centers of the pancakes start to bubble.


Heat Griddle
Measure Flour
Beat Egg
Measure Sugar
Oil Griddle
Combine Dry Ingredients
Make Pancakes
Serve Pancakes
Add Wet Ingredients
Measure Baking Pwdr
Melt Butter
Measure Salt
Measure Milk

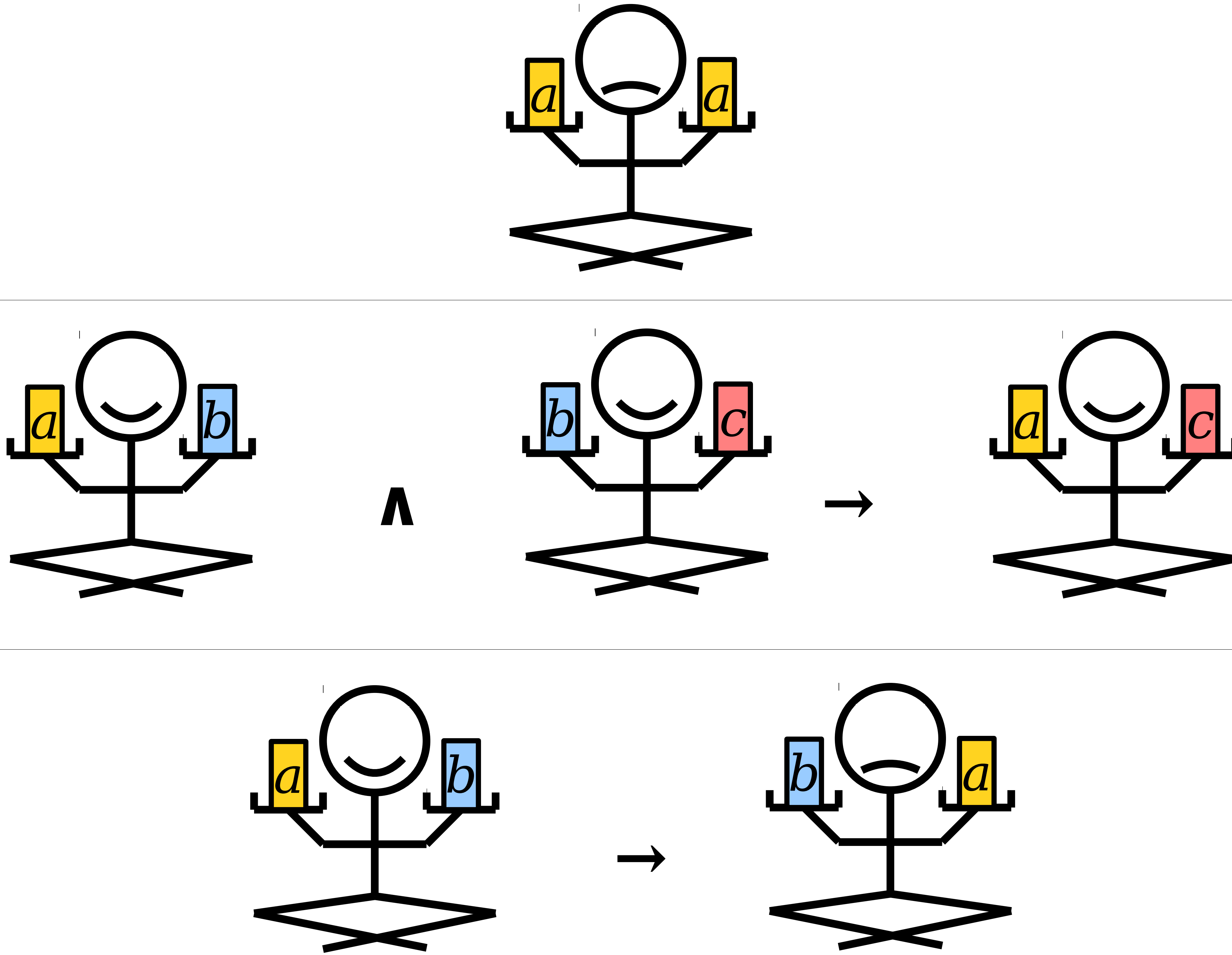
 

Relations and Prerequisites

- Let's imagine that we have a prerequisite structure with no circular dependencies.
- We can think about a binary relation R where aRb means

“***a* must happen before *b***”

- What properties of R could we deduce just from this?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$$\forall a \in A.\ a \not R a$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not R a)$$

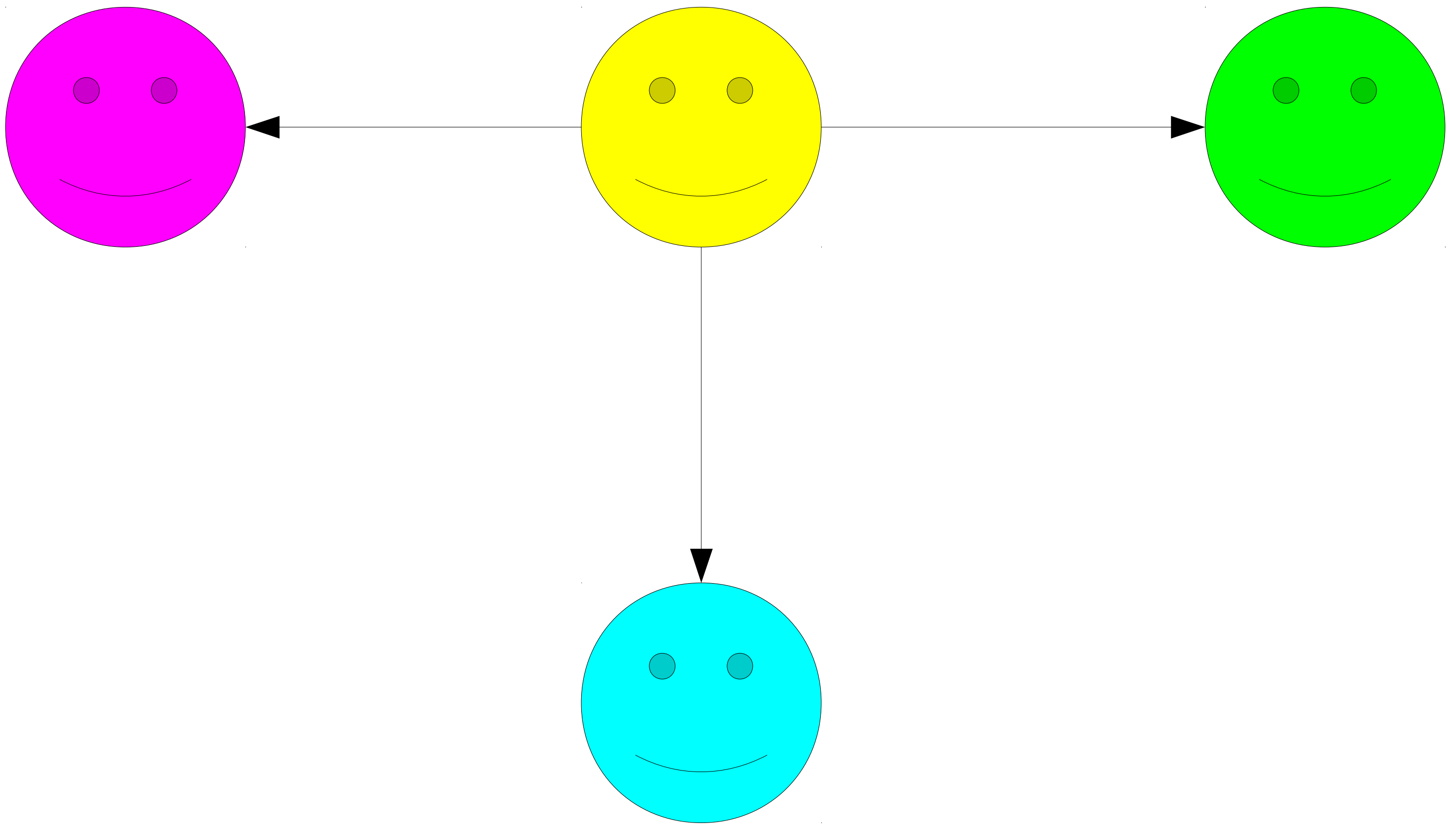
Irreflexivity

- Some relations *never* hold from any element to itself.
- As an example, $x \not< x$ for any x.
- Relations of this sort are called ***irreflexive***.
- Formally speaking, a binary relation R over a set A is irreflexive if the following first-order logic statement is true about R:

$$\forall a \in A.\ a \not R a$$

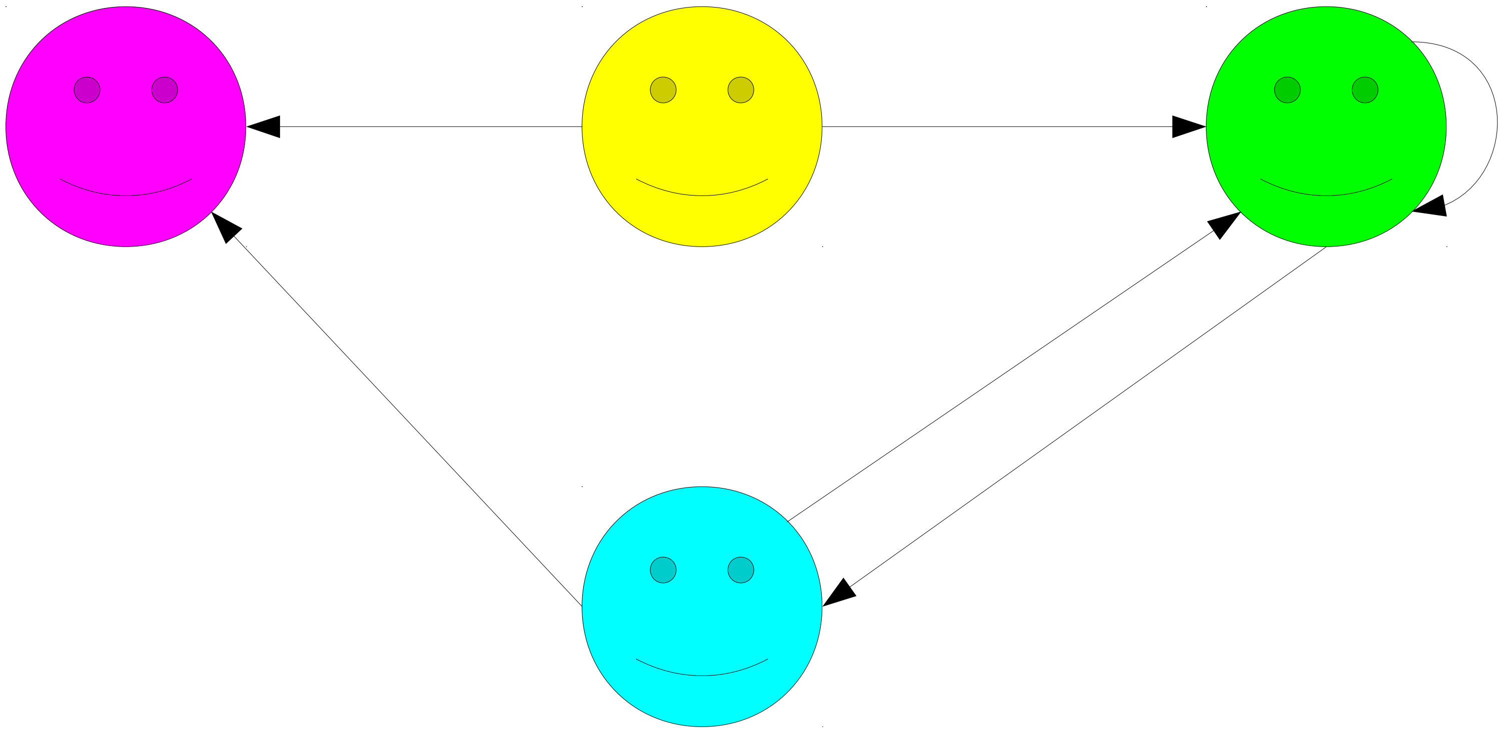
(*"No element is related to itself."*)

Irreflexivity Visualized



$$\forall a \in A.\ a \not R a$$

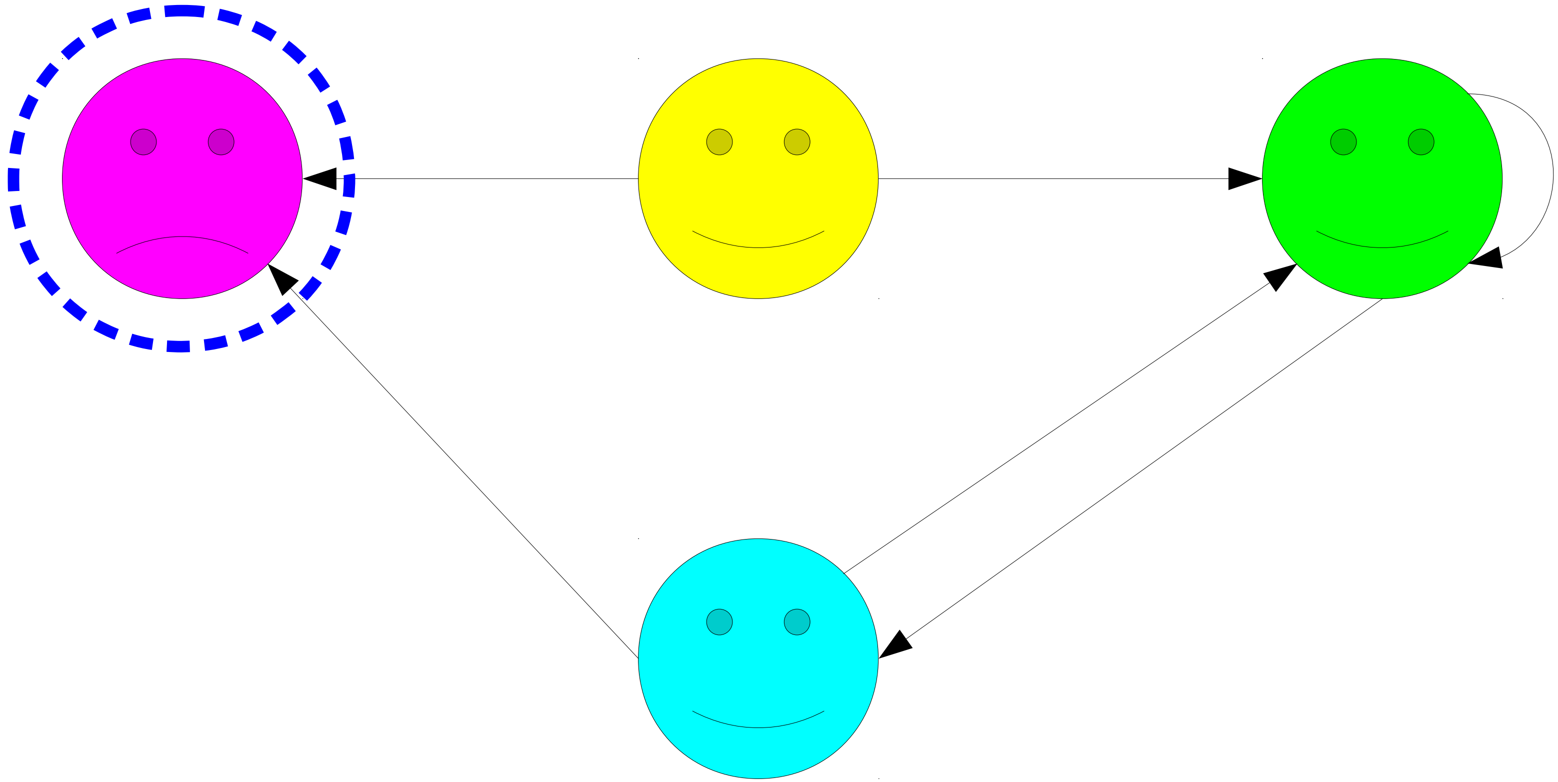
(*"No element is related to itself."*)



Let R be the relation depicted here.
How many of the following claims are true?

R is reflexive.
R is not reflexive.
R is irreflexive.
R is not irreflexive.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

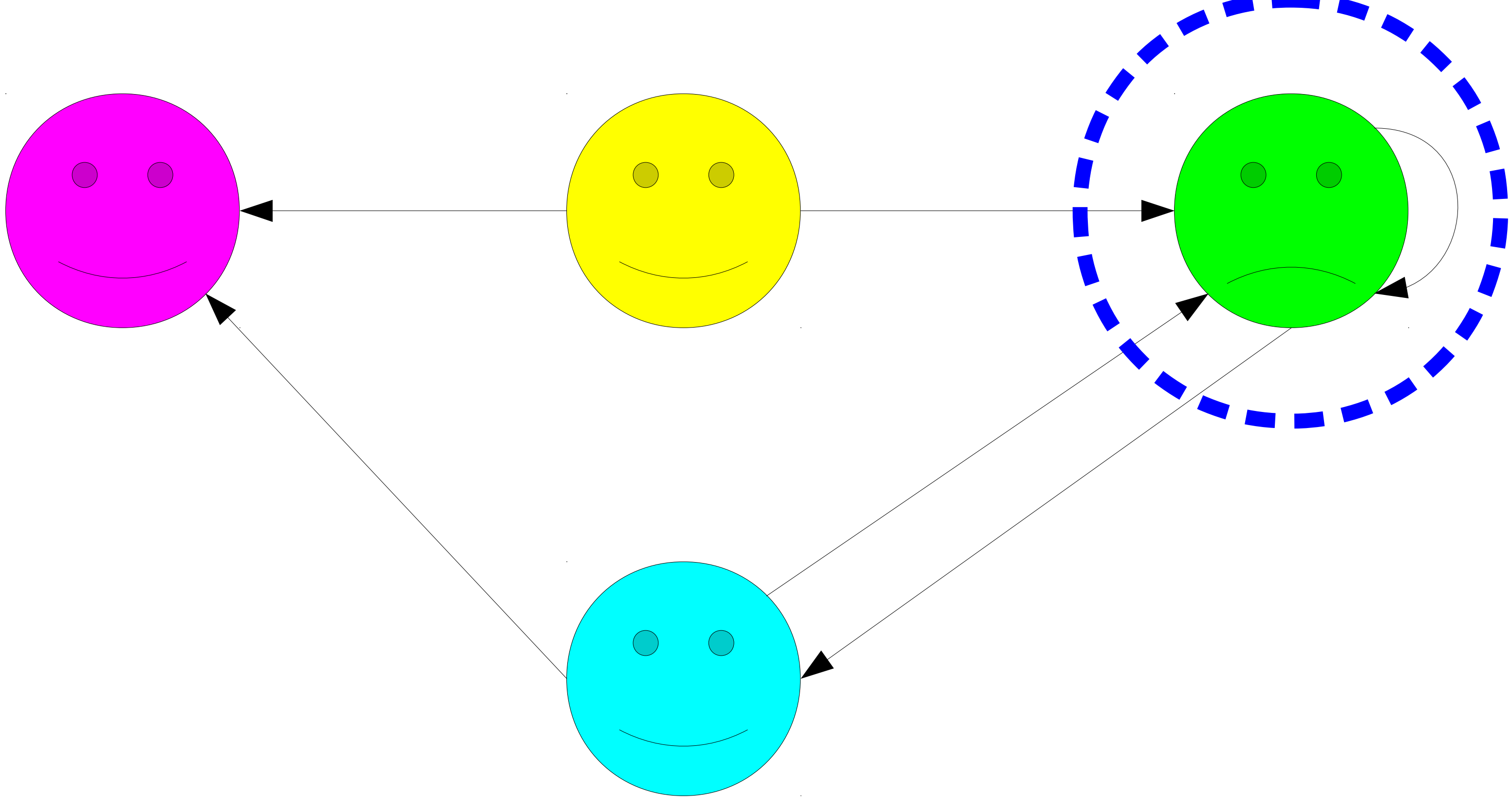


Is this relation reflexive?

Nope!

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")



Is this relation irreflexive?

$$\forall a \in A.\ a \not{R} a$$

(*"No element is related to itself."*)

Reflexivity and Irreflexivity

- Reflexivity and irreflexivity are ***not*** opposites!
- Here's the definition of reflexivity:

$$\forall a \in A.\ aRa$$

- What is the negation of the above statement?

$$\exists a \in A.\ a \not R a$$

- What is the definition of irreflexivity?

$$\forall a \in A.\ a \not R a$$

$$\forall a \in A.\ a \not{R} a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not{R} a)$$

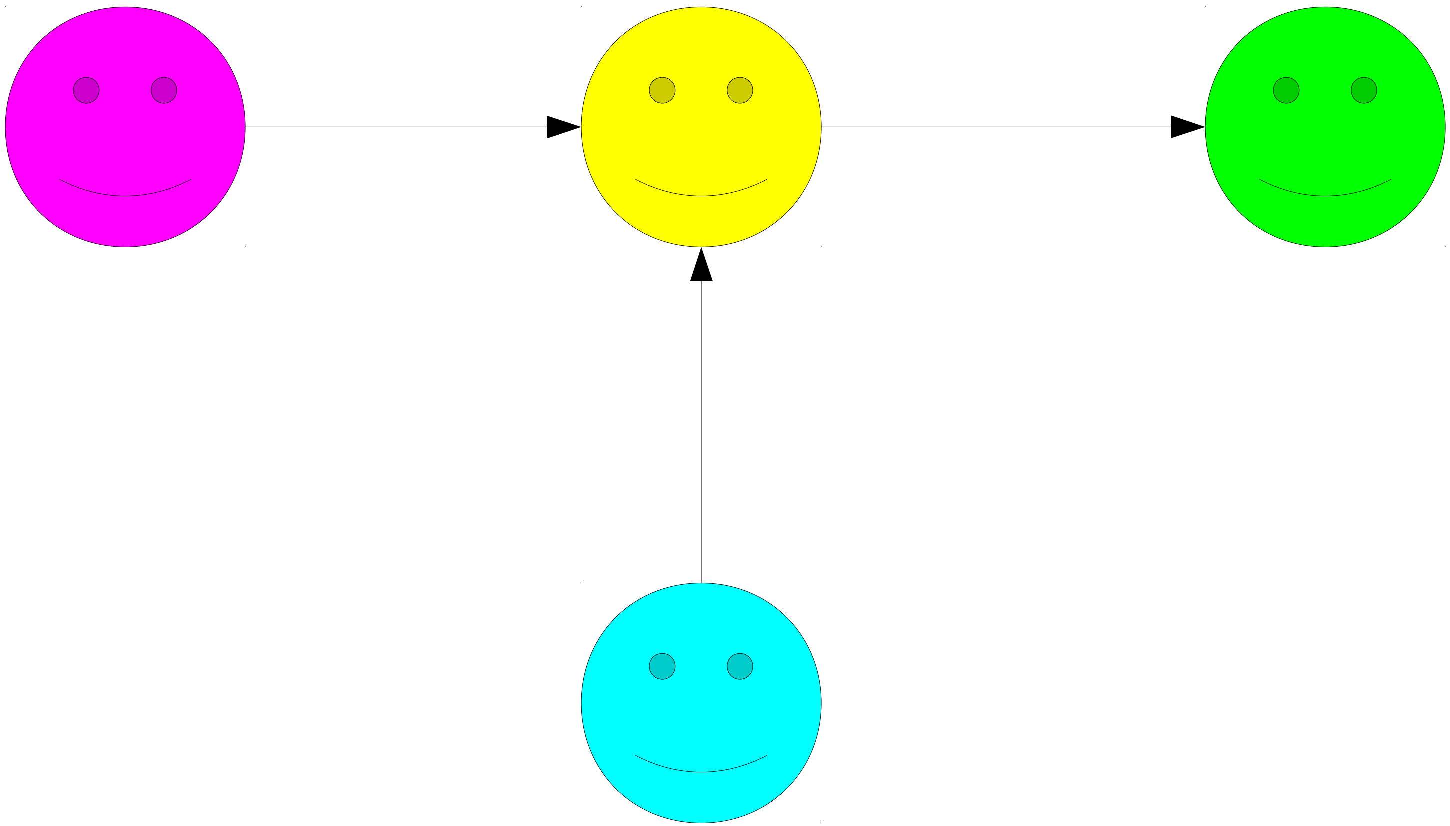
Asymmetry

- In some relations, the relative order of the objects can never be reversed.
- As an example, if $x < y$, then $y \not< x$.
- These relations are called ***asymmetric***.
- Formally: a binary relation R over a set A is called *asymmetric* if the following first-order logic statement is true about R:

$$\forall a \in A. \forall b \in A. (aRb \to b \not{R} a)$$

("If a relates to b, then b does not relate to a.")

Asymmetry Visualized



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

(*"If a relates to b, then b does not relate to a."*)

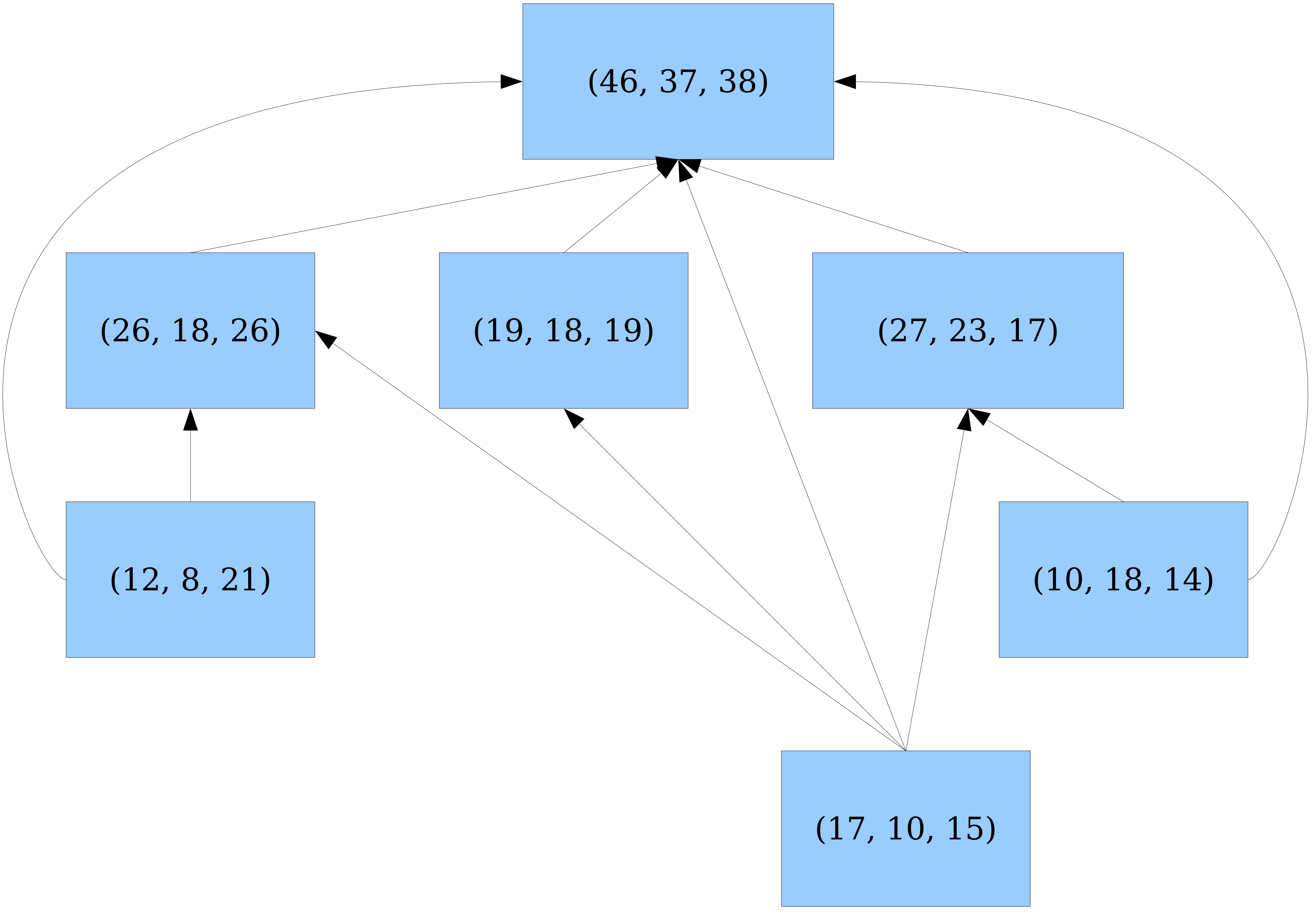
Question to Ponder: Are symmetry and asymmetry opposites of one another?

Strict Orders

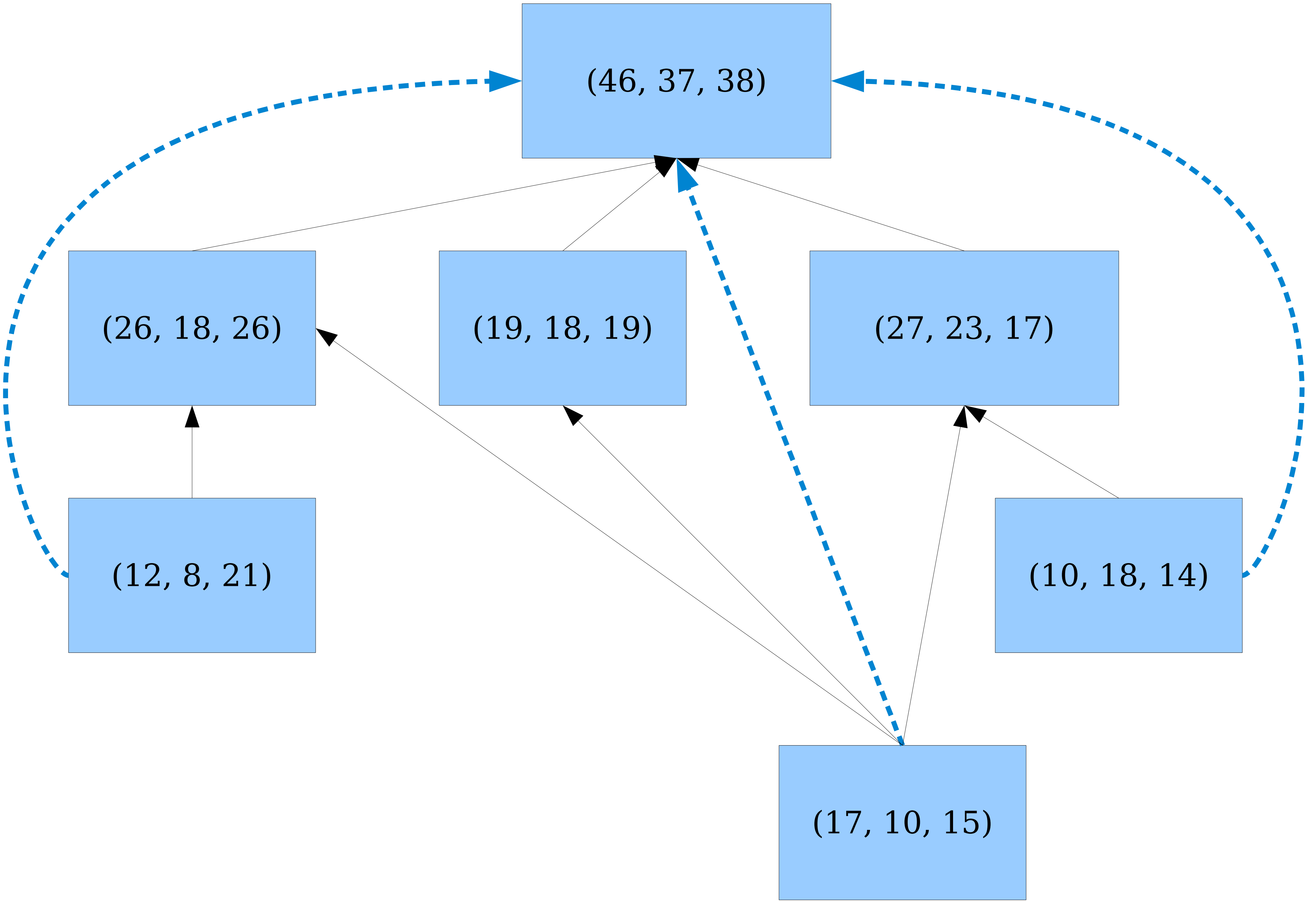
- A ***strict order*** is a relation that is irreflexive, asymmetric and transitive.
- Some examples:
  - $x < y$.
  - a can run faster than b.
  - $A \subsetneq B$ (that is, $A \subseteq B$ and $A \neq B$).
- Strict orders are useful for representing prerequisite structures and have applications in complexity theory (measuring notions of relative hardness) and algorithms (searching and sorting).

Drawing Strict Orders

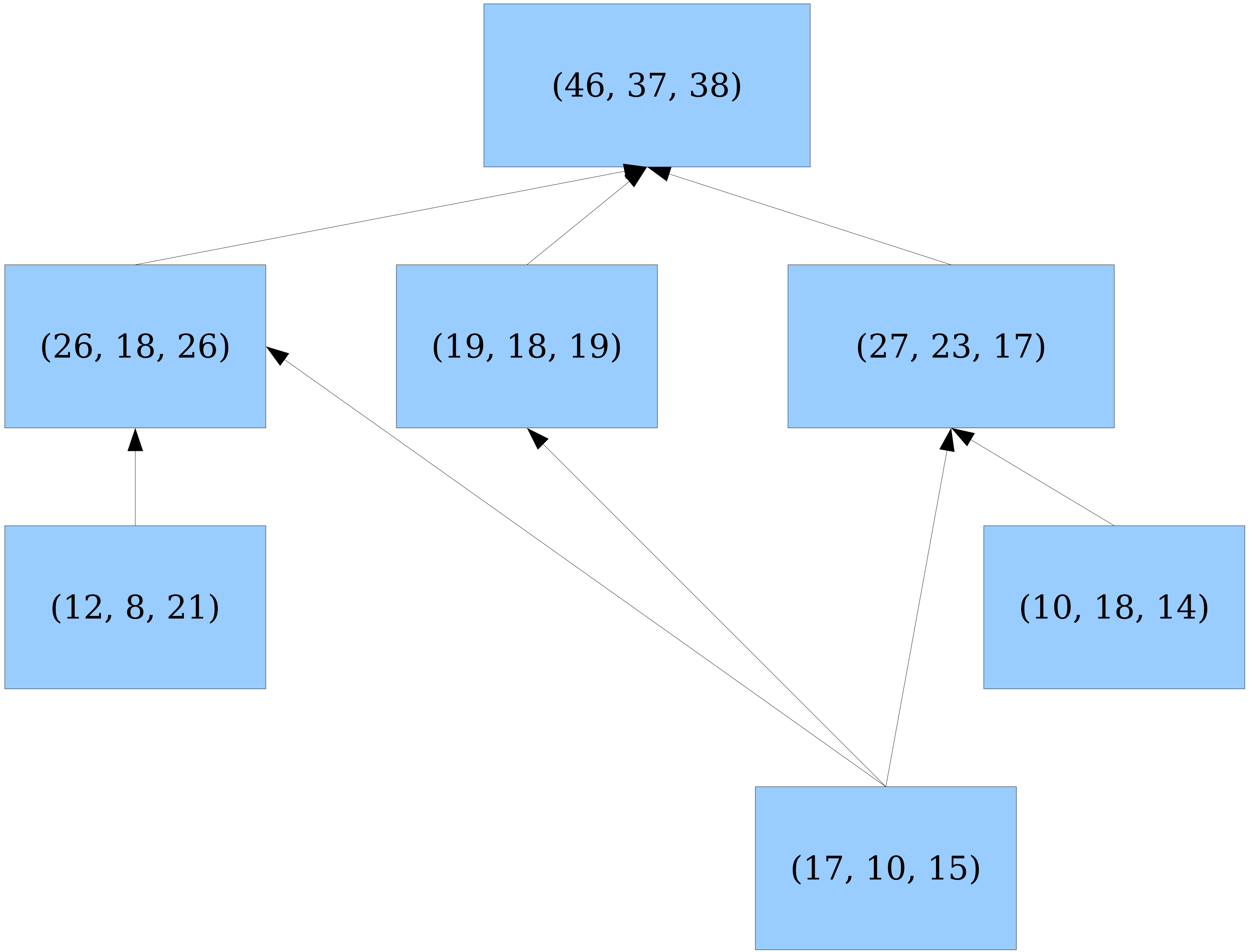
Gold	Silver	Bronze
46	37	38
27	23	17
26	18	26
19	18	19
17	10	15
12	8	21
10	18	14
9	3	9
8	12	8
8	11	10
8	7	4
8	3	4
7	6	6
7	4	6
6	6	1
6	3	2



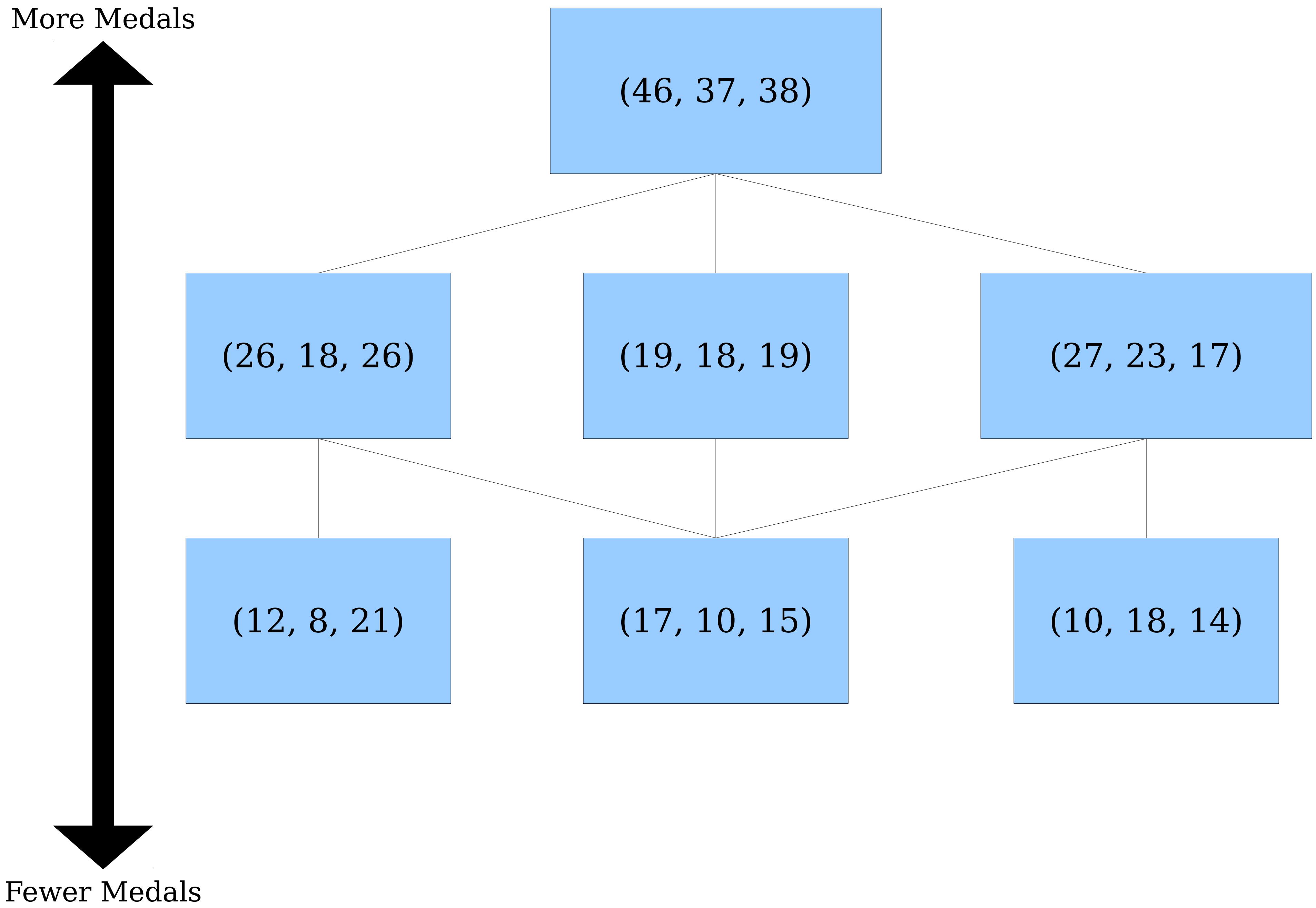
$(g_1, s_1, b_1)\ R\ (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \ \wedge\ s_1 < s_2 \ \wedge\ b_1 < b_2$



$$(g_1, s_1, b_1)\ R\ (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \ \wedge\ s_1 < s_2 \ \wedge\ b_1 < b_2$$



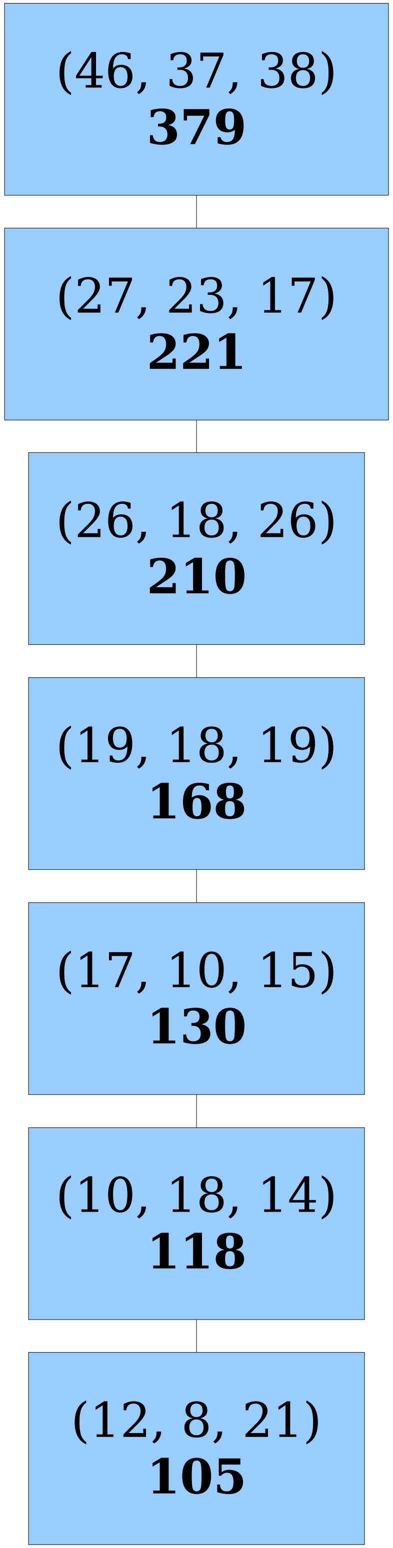
$$(g_1, s_1, b_1)\ R\ (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \ \wedge\ s_1 < s_2 \ \wedge\ b_1 < b_2$$



$$(g_1, s_1, b_1)\ R\ (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \ \land \ s_1 < s_2 \ \land \ b_1 < b_2$$

Hasse Diagrams

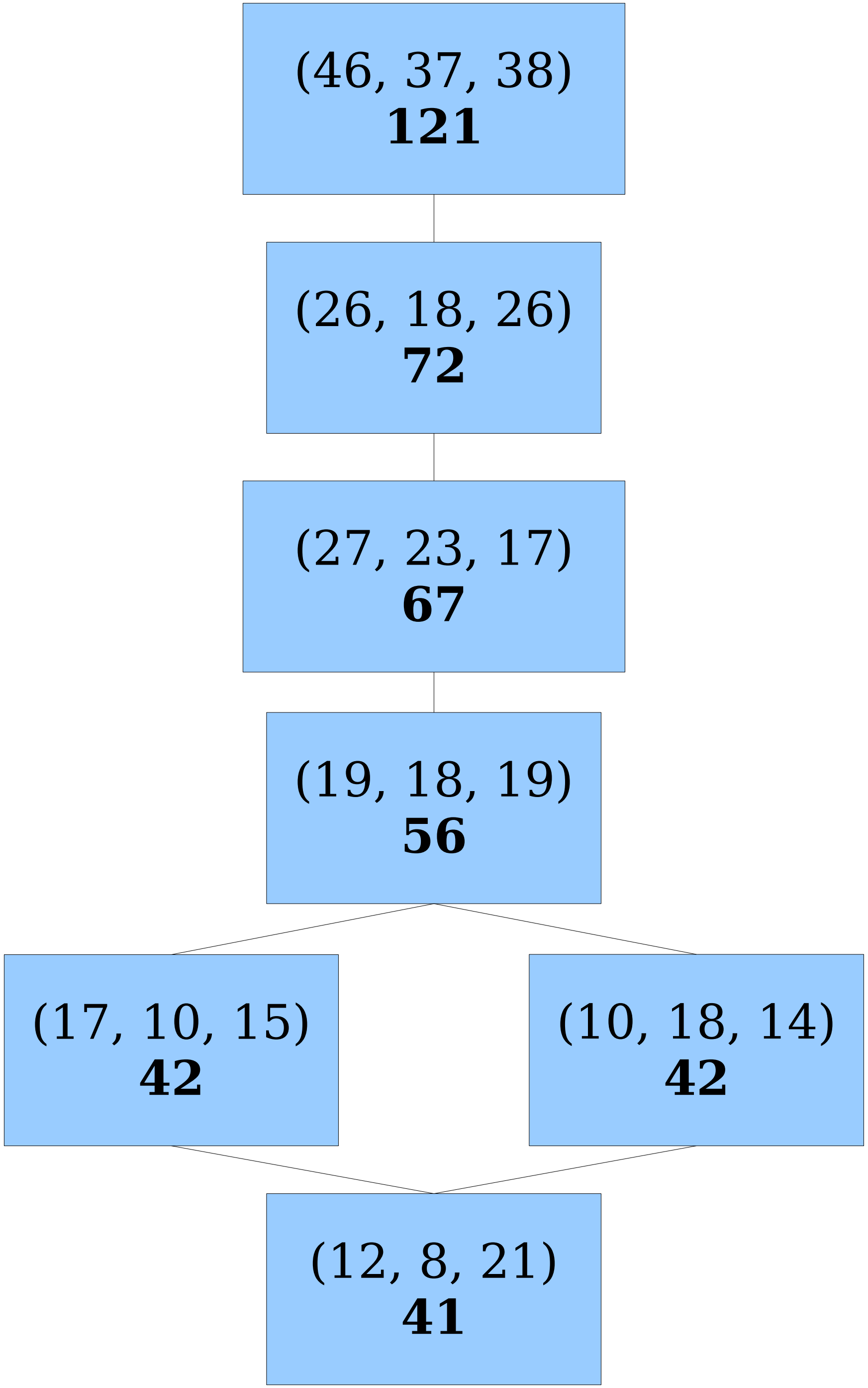
- A ***Hasse diagram*** is a graphical representation of a strict order.
- Elements are drawn from bottom-to-top.
- No self loops are drawn, and none are needed! By ***irreflexivity*** we know they shouldn't be there.
- Higher elements are bigger than lower elements: by ***asymmetry***, the edges can only go in one direction.
- No redundant edges: by ***transitivity***, we can infer the missing edges.



$(g_1, s_1, b_1)\ T\ (g_2, s_2, b_2)$

if

$5g_1 + 3s_1 + b_1 < 5g_2 + 3s_2 + b_2$

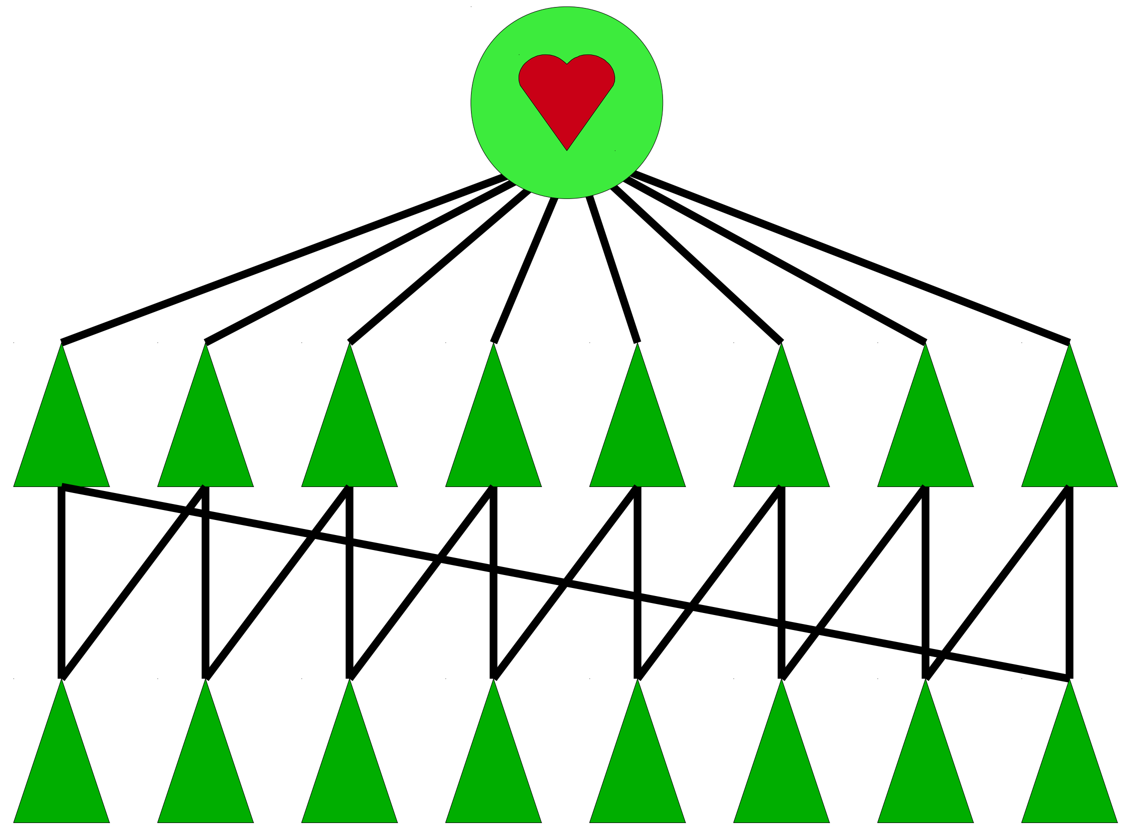


$$(g_1, s_1, b_1)\ U\ (g_2, s_2, b_2)$$

if

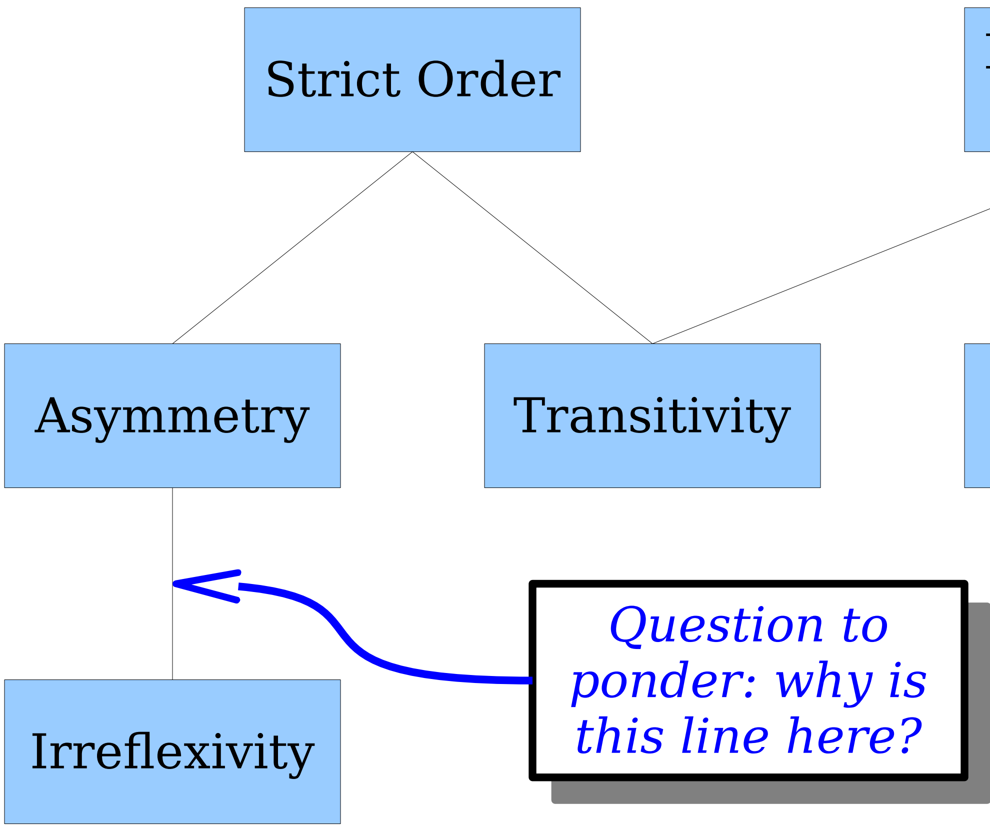
$$g_1 + s_1 + b_1 < g_2 + s_2 + b_2$$

Hasse Artichokes



xRy if x must be eaten before y

The Meta Strict Order



aRb **if** ***a* is less specific than *b***

Next Time

- ***Functions***
  - How do we model transformations in a mathematical sense?
- ***Domains and Codomains***
  - Type theory meets mathematics!
- ***Injections, Surjections, and Bijections***
  - Three special classes of functions.