Cardinality
Recap from Last Time
Domains and Codomains

- Every function $f$ has two sets associated with it: its \textit{domain} and its \textit{codomain}.
- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.
- We write $f : A \rightarrow B$ to indicate that $f$ is a function whose domain is $A$ and whose codomain is $B$.

The function must be defined for each element of its domain.

The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.
Function Composition

• If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, the composition of $f$ and $g$, denoted $g \circ f$, is a function
  • whose domain is $A$,
  • whose codomain is $C$, and
  • which is evaluated as $(g \circ f)(x) = g(f(x))$. 
Injective Functions

• A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if different inputs always map to different outputs.
  • A function with this property is called an **injection**.
  • Formally, $f : A \rightarrow B$ is an injection if this FOL statement is true:

    $$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

    (“If the inputs are different, the outputs are different”)

  • Equivalently:

    $$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

    (“If the outputs are the same, the inputs are the same”)

• **Theorem:** The composition of two injections is an injection.
Surjective Functions

- A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if each element of the codomain is “covered” by at least one element of the domain.
  - A function with this property is called a **surjection**.
  - Formally, $f : A \rightarrow B$ is a surjection if this FOL statement is true:

$$\forall b \in B. \exists a \in A. f(a) = b$$

(“For every possible output, there's at least one possible input that produces it”)

- **Theorem:** The composition of two surjections is a surjection.
Bijectons

• A function that associates each element of the codomain with a unique element of the domain is called **bijective**.
  • Such a function is a **bijection**.

• Formally, a bijection is a function that is both **injective** and **surjective**.

• **Theorem:** The composition of two bijections is a bijection.
New Stuff!
Cardinality Revisited
Cardinality

- Recall *(from our first lecture!)* that the **cardinality** of a set is the number of elements it contains.

- If \( S \) is a set, we denote its cardinality by \(|S|\).

- For finite sets, cardinalities are natural numbers:
  - \(|\{1, 2, 3\}| = 3\)
  - \(|\{100, 200\}| = 2\)

- For infinite sets, we introduced **infinite cardinals** to denote the size of sets:
  \[ |\mathbb{N}| = \aleph_0 \]
Defining Cardinality

• It is difficult to give a rigorous definition of what cardinalities actually are.
  • What is 4? What is $\aleph_0$?
  • (Take Math 161 for an answer!)

• **Idea:** Define cardinality as a relation between two sets rather than an absolute quantity.
Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

\[ |S| = |T| \text{ if there exists a bijection } f : S \to T \]
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Fun with Cardinality
Terminology Refresher

• Let $a$ and $b$ be real numbers where $a \leq b$.

• The notation $[a, b]$ denotes the set of all real numbers between $a$ and $b$, inclusive.
\[
[a, b] = \{ x \in \mathbb{R} | a \leq x \leq b \}
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• The notation $(a, b)$ denotes the set of all real numbers between $a$ and $b$, exclusive.
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(a, b) = \{ x \in \mathbb{R} | a < x < b \}
\]
Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?
Home on the Range
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\[ f : [0, 1] \rightarrow [0, 2] \]

\[ f(x) = 2x \]
**Theorem:** \(|[0, 1]| = |[0, 2]|\)

**Proof:**
Consider the function \(f : [0, 1] \rightarrow [0, 2]\) defined as \(f(x) = 2x\).
We will prove that \(f\) is a bijection.

First, we will show that \(f\) is a well-defined function. Choose any \(x \in [0, 1]\). This means that \(0 \leq x \leq 1\), so we know that \(0 \leq 2x \leq 2\). Consequently, we see that \(0 \leq f(x) \leq 2\), so \(f(x) \in [0, 2]\).

Next, we'll show that \(f\) is injective. Pick any \(x_1, x_2 \in [0, 1]\) where \(f(x_1) = f(x_2)\). We will show that \(x_1 = x_2\). To see this, notice that since \(f(x_1) = f(x_2)\), we see that \(2x_1 = 2x_2\), which in turn tells us \(x_1 = x_2\), as required.

Finally, we will show that \(f\) is surjective. To do so, consider any \(y \in [0, 2]\). We'll show that there is some \(x \in [0, 1]\) where \(f(x) = y\).
Let \(x = \frac{y}{2}\). Since \(y \in [0, 2]\), we know \(0 \leq y \leq 2\), and therefore that \(0 \leq \frac{y}{2} \leq 1\). We picked \(x = \frac{y}{2}\), so we know that \(0 \leq x \leq 1\), which in turn means \(x \in [0, 1]\). Moreover, notice that \(f(x) = 2x = 2\left(\frac{y}{2}\right) = y\), so \(f(x) = y\), as required. ■
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Let $x = y/2$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq y/2 \leq 1$. We picked $x = y/2$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 2x = 2(y/2) = y,$$

so $f(x) = y$, as required.
**Theorem:** $|[0, 1]| = |[0, 2]|$

**Proof:** Consider the function $f : [0, 1] \rightarrow [0, 2]$ defined as $f(x) = 2x$. We will prove that $f$ is a bijection.

First, we will show that $f$ is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \leq x \leq 1$, so we know that $0 \leq 2x \leq 2$. Consequently, we see that $0 \leq f(x) \leq 2$, so $f(x) \in [0, 2]$.

Next, we’ll show that $f$ is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that $f$ is surjective. To do so, consider any $y \in [0, 2]$. We’ll show that there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = \frac{y}{2}$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq \frac{y}{2} \leq 1$. We picked $x = \frac{y}{2}$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 2x = 2(\frac{y}{2}) = y,$$

so $f(x) = y$, as required. ■
**Theorem:** \(|[0, 1]| = |[0, 2]|\)

**Proof:** Consider the function \(f : [0, 1] \to [0, 2]\) defined as \(f(x) = 2x\). We will prove that \(f\) is a bijection.

First, we will show that \(f\) is a well-defined function. Choose any \(x \in [0, 1]\). This means that \(0 \leq x \leq 1\), so we know that \(0 \leq 2x \leq 2\). Consequently, we see that \(0 \leq f(x) \leq 2\), so \(f(x) \in [0, 2]\).

Next, we’ll show that \(f\) is injective. Pick any \(x_1, x_2 \in [0, 1]\) where \(f(x_1) = f(x_2)\). We will show that \(x_1 = x_2\). Since \(f(x_1) = f(x_2)\), we see that \(2x_1 = 2x_2\), which tells us that \(x_1 = x_2\), as required.

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Let \(x = y/2\). Since \(y \in [0, 2]\), we know that \(0 \leq y \leq 2\), and therefore that \(0 \leq y/2 \leq 1\). We picked \(x = y/2\), so we know that \(0 \leq x \leq 1\), which in turn means \(x \in [0, 1]\). Moreover, notice that

\[
f(x) = 2x = 2(y/2) = y,
\]

so \(f(x) = y\), as required. 

When defining something we claim is a function, the convention is to prove that it obeys the domain/codomain rules. For whatever reason, there isn’t a convention of showing that it’s deterministic. Ah, tradition. 😃
**Theorem:** $|[0, 1]| = |[0, 2]|$

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$$f(x) = 2x = 2(y/2) = y,$$

so $f(x) = y$, as required. ■
Home on the Range

\[ f : [0, 1] \rightarrow [0, 2] \]

\[ f(x) = 2x \]
Home on the Range

\[ f : [0, 1] \to [0, 3] \]
\[ f(x) = 3x \]
Home on the Range

\[ f : [0, 1] \rightarrow [0, 137] \]

\[ f(x) = 137x \]
This means that cardinality (how many points there are) is a different idea than mass (how much those points weight). Look into measure theory if you're curious to learn more!
And one more example, just for funzies.
\[ f : (-\pi/2, \pi/2) \rightarrow \mathbb{R} \]
\[ f(x) = \tan x \]
\[ |(-\pi/2, \pi/2)| = |\mathbb{R}| \]
Some Properties of Cardinality
Theorem: For any set $A$, we have $|A| = |A|$. 
**Theorem:** For any set \( A \), we have \(|A| = |A|\).

**Proof:**
**Theorem:** For any set $A$, we have $|A| = |A|$.

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**Theorem:** For any set $A$, we have $|A| = |A|$.  

**Proof:** Consider any set $A$, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that $f$ is a bijection.
**Theorem:** For any set \( A \), we have \( |A| = |A| \).

**Proof:** Consider any set \( A \), and let \( f : A \to A \) be the function defined as \( f(x) = x \). We will prove that \( f \) is a bijection.

First, we’ll show that \( f \) is a well-defined function.
**Theorem:** For any set $A$, we have $|A| = |A|$.

**Proof:** Consider any set $A$, and let $f : A \rightarrow A$ be the function defined as $f(x) = x$. We will prove that $f$ is a bijection.

First, we’ll show that $f$ is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.
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**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
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Proof: Consider any sets $A$, $B$, and $C$ where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from $A$ to $C$. 

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Since $|A| = |B|$, we know that there is a some bijection $f : A \rightarrow B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \rightarrow C$. 
**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Since $|A| = |B|$, we know that there is a some bijection $f : A \rightarrow B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \rightarrow C$.

Consider the function $g \circ f : A \rightarrow C$. 
**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

**Proof:** Consider any sets $A$, $B$, and $C$ where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from $A$ to $C$.

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Consider the function $g \circ f : A \rightarrow C$. Since $g$ and $f$ are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from $A$ to $C$. 
**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

**Proof:** Consider any sets $A$, $B$, and $C$ where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from $A$ to $C$.

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Consider the function $g \circ f : A \to C$. Since $g$ and $f$ are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from $A$ to $C$. Thus $|A| = |C|$, as required.
**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

**Proof:** Consider any sets $A$, $B$, and $C$ where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from $A$ to $C$.

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Consider the function $g \circ f : A \rightarrow C$. Since $g$ and $f$ are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from $A$ to $C$. Thus $|A| = |C|$, as required. ■
Great exercise: Prove that if $A$ and $B$ are sets where $|A| = |B|$, then $|B| = |A|$.
Time-Out for Announcements!
RSVP FOR

ASK-ME-ANYTHING

with Professor Nick McKeown
on the internet, abolishing the death penalty, and planning your journey

Wednesday, October 16th
6:00pm - 7:00pm | Gates 104
Dinner will be served!

RSVP using this link by midnight tonight!
Midterm Exam Logistics

- Our first midterm exam is next **Monday, October 21st**, from **7:00PM - 10:00PM**. Locations are divvied up by last (family) name:
  - A – G: Go to Hewlett 201.
- You’re responsible for Lectures 00 – 05 and topics covered in PS1 – PS2. Later lectures (relations forward) and problem sets (PS3 onward) won’t be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5” × 11” sheet of notes with you to the exam, decorated however you’d like.
- Students with OAE accommodations: we will be reaching out to you soon with room and time assignments. Please contact us **immediately** if we don’t yet have your OAE letter.
Midterm Exam

- **We want you to do well on this exam.** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.

- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your “mathematical potential” or “innate mathematical ability.”
Preparing for the Exam
Learn by doing.

Learn by reading.
Extra Practice Problems

• Up on the course website, you’ll find
  • Extra Practice Problems 1 (a set of cumulative review problems), and
  • three practice midterm exams, each of which is a (slightly modified) version of a real exam we’ve given out in a previous quarter.
• Take the time to work through some of these problems. This is, perhaps, the best way to study.
You can always run your code and just see what happens!

Checking a proof requires human expertise.

Rapid iteration. Constant, small feedback.

Slower iteration. Infrequent, large feedback.
Practice Midterm Exam

• To help you prepare for the midterm, we'll be holding a practice midterm exam **Wednesday** from **7PM - 10PM** in **Cemex Auditorium**.
  
  • The exam we’ll use isn’t one of the ones posted up on the course website, so feel free to use those as practice in the meantime.

• Course staff will be on hand to answer your questions.

• Can't make it? We'll release that practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!
Doing Practice Problems

- As you work through practice problems, *keep other humans in the loop!*
- Ask your problem set partner to review your answers and offer feedback – and volunteer to do the same!
- Post your answers as private questions on Piazza and ask for TA feedback!
- *Feedback loops are key to improving!"
Preparing for the Exam

• We've released a handout (Handout 19) containing advice about how to prepare for the exam, along with advice from previous CS103 students.

• Read over it... there's good advice there!
Your Questions
“What advice do you have for somebody who’s going through an academic and social slump? Have you ever experienced slumps, and if so, how did you deal with them?”

For starters, I hope everything is okay! I wish I had a “one-size-fits-all” answer to this question, but it’s going to depend on the root causes. For example, if you just had a rough breakup, I’d offer totally different advice than if you’re feeling burnt out from working too hard and taking on too many commitments.

One piece of advice that works well across these: think about what sorts of things do you need to do to recharge and feel better. For some people that’s exercise, for others it’s socializing, for others it’s taking a long walk, for others it’s meditating, etc. Make an effort to figure out what activities leave you feeling energized, and make sure to take time for them. Speaking from experience, this makes a huge difference.

For everyone else – check in with folks around you who don’t seem to be doing all that great. It makes a world of difference.
“I really like Computer Science, but sometimes I don’t feel as competent as everyone else here/ I feel slow getting concepts. Any advice to combat this feeling?”

A few things to keep in mind:

1. Don’t mistake talent for experience.
2. Don’t mistake unions for intersections.
3. Don’t perceive more scarcity than there is.
4. Don’t discount the happiness you’ve found.
Back to CS103!
Unequal Cardinalities

• Recall: $|A| = |B|$ if the following statement is true:

  There exists a bijection $f : A \to B$

• What does it mean for $|A| \neq |B|$ to be true?

  Every function $f : A \to B$ is not a bijection.

• This is a strong statement! To prove $|A| \neq |B|$, we need to show that no possible function from $A$ to $B$ can be injective and surjective.
Unequal Cardinalities

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Cantor’s Theorem Revisited
Cantor’s Theorem

- In our very first lecture, we sketched out a proof of Cantor’s theorem, which says that

  If $S$ is a set, then $|S| < |\mathcal{P}(S)|$.

- That proof was visual and pretty hand-wavy. Let’s see if we can go back and formalize it!
Where We’re Going

• Today, we’re going to formally prove the following result:

\[
\text{If } S \text{ is a set, then } |S| \neq |\mathcal{P}(S)|.
\]

• We’ve released an online Guide to Cantor’s Theorem, which will go into way more depth than what we’re going to see here.

• The goal for today will be to see how to start with our picture and turn it into something rigorous.

• On the next problem set, you’ll explore the proof in more depth and see some other applications.
The Roadmap

• We’re going to prove this statement:
  If $S$ is a set, then $|S| \neq |\wp(S)|$.

• Here’s how this will work:
  • Pick an arbitrary set $S$.
  • Pick an arbitrary function $f : S \to \wp(S)$.
  • Show that $f$ is not surjective using a diagonal argument.
  • Conclude that there are no bijections from $S$ to $\wp(S)$.
  • Conclude that $|S| \neq |\wp(S)|$. 
The Roadmap

We’re going to prove this statement:

If \( S \) is a set, then \(|S| \neq |\wp(S)|\).

Here’s how this will work:

- Pick an arbitrary set \( S \).
- Pick an arbitrary function \( f : S \to \wp(S) \).
- **Show that \( f \) is not surjective using a diagonal argument.**

Conclude that there are no bijections from \( S \) to \( \wp(S) \).
Conclude that \(|S| \neq |\wp(S)|\).
This is a drawing of our function \( f : S \to \mathcal{P}(S) \).

\[
\begin{align*}
  x_0 & \leftrightarrow \left\{ x_0, x_2, x_4, \ldots \right\} \\
  x_1 & \leftrightarrow \left\{ x_3, x_5, \ldots \right\} \\
  x_2 & \leftrightarrow \left\{ x_0, x_1, x_2, x_5, \ldots \right\} \\
  x_3 & \leftrightarrow \left\{ x_1, x_4, \ldots \right\} \\
  x_4 & \leftrightarrow \left\{ x_2, \ldots \right\} \\
  x_5 & \leftrightarrow \left\{ x_0, x_4, x_5, \ldots \right\} \\
  \ldots & \leftrightarrow \left\{ \ldots \right\}
\end{align*}
\]
This is a drawing of our function $f : S \to \mathcal{P}(S)$.

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<td>{ $x_0, x_4, x_5, \ldots$ }</td>
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This is a drawing of our function $f : S \to \mathcal{P}(S)$. 

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This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

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</tbody>
</table>

Which element is paired with this set?
This is a drawing of our function \( f : S \rightarrow \mathcal{P}(S) \).

\[
\begin{array}{cccccc}
X_0 & X_1 & X_2 & X_3 & X_4 & X_5 & \ldots \\
\{ X_0, & X_2, & X_4, & \ldots \} & \{ X_3, & X_5, & \ldots \} & \{ X_0, & X_2, & X_5, & \ldots \} & \{ X_1, & X_4, & \ldots \} & \{ X_2, & \ldots \} & \{ X_0, & X_4, & X_5, & \ldots \} & \{ \ldots \} \\
\{ \ldots \} & \{ \ldots \} & \{ \ldots \} & \{ \ldots \} & \{ \ldots \} & \{ \ldots \} & \{ \ldots \}
\end{array}
\]

"Flip" this set. Swap what’s included and what’s excluded.

\[
\{ X_1, X_3, X_4, \ldots \}
\]
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<tr>
<th></th>
<th>X₀</th>
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<th>X₂</th>
<th>X₃</th>
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</table>

This is a drawing of our function \( f : S \to \mathcal{P}(S) \).

Which element is paired with this set?
This is a drawing of our function \( f : S \to \mathcal{P}(S) \).

Which element is paired with this set?
Which element is paired with this set?

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Which element is paired with this set?
This is a drawing of our function \( f : S \to \mathcal{P}(S) \).

Which element is paired with this set?
This is a drawing of our function $f : S \rightarrow \wp(S)$.

Which element is paired with this set?

|   | $X_0$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | ...
|---|-------|-------|-------|-------|-------|-------|-------
|$X_0$ | $\{X_0, X_2, X_4, \ldots\}$ |       |       |       |       |       |       
|$X_1$ |       | $\{X_1, X_2, X_5, \ldots\}$ |       |       |       |       |       
|$X_2$ |       |       | $\{X_0, X_2, X_5, \ldots\}$ |       |       |       |       
|$X_3$ |       |       |       | $\{X_1, X_4, \ldots\}$ |       |       |       
|$X_4$ |       |       |       |       | $\{X_2, X_4, \ldots\}$ |       |       
|$X_5$ |       |       |       |       |       | $\{X_0, X_4, X_5, \ldots\}$ |       
|$\ldots$ |       |       |       |       |       |       | $\{\ldots, \ldots, \ldots, \ldots\}$ |
This is a drawing of our function \( f : S \to \mathcal{P}(S) \).

Which element is paired with this set?

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<td>{X₀, X₂, X₄, …}</td>
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</table>
What set is this?

This is a drawing of our function \( f : S \to \wp(S) \).
This is a drawing of our function $f: S \rightarrow \wp(S)$.

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<td>$X_0$</td>
<td>${X_0, X_1, X_4, \ldots}$</td>
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$X_1 \notin f(X_1)$

Why is $X_1$ in this set?
This is a drawing of our function $f : S \to \mathcal{P}(S)$.

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$\not\in f(x_3)$

Why is $x_3$ in this set?
Why is $x_4$ in this set?

This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.
Why isn’t $x_0$ in this set?

This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

- $x_0 \in f(x_0)$

- $f(x_0)$

- Why isn’t $x_0$ in this set?
Why isn't $x_2$ in this set?

This is a drawing of our function $f : S \to \wp(S)$. 

$x_2 \in f(x_2)$
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This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

Why isn't $X_5$ in this set? 

$X_5 \in f(X_5)$

$f(X_5)$
This is a drawing of our function $f : S \to \wp(S)$.

If $x \not\in f(x)$, include $x$ in the set.
If $x \in f(x)$, exclude $x$ from the set.
We have a function $f : S \to \mathcal{P}(S)$. If $x \notin f(x)$, include $x$ in the set. If $x \in f(x)$, exclude $x$ from the set. Define $D = \{ x \in S \mid x \notin f(x) \}$.

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The Diagonal Set

- For any set $S$ and function $f : S \to \wp(S)$, we can define a set $D$ as follows:

  \[ D = \{ x \in S \mid x \notin f(x) \} \]

  ("The set of all elements $x$ where $x$ is not an element of the set $f(x)$.")

- This is a formalization of the set we found in the previous picture.

- Using this choice of $D$, we can formally prove that no function $f : S \to \wp(S)$ is a bijection.
**Theorem:** If $S$ is a set, then $|S| \neq |\mathcal{P}(S)|$. 

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\mathcal{P}(S)|$ by showing that there are no bijections from $S$ to $\mathcal{P}(S)$. To do so, choose an arbitrary function $f : S \to \mathcal{P}(S)$. We will prove that $f$ is not surjective. 

Starting with $f$, we define the set $D = \{ x \in S | x \notin f(x) \}$. 

(1) We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. By the definition of $D$, we know that $y \in D$ iff $y \notin f(y)$. 

(2) By assumption, $f(y) = D$. Combined with (2), this tells us $y \in D$ iff $y \notin D$. 

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The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor’s theorem can be formalized in terms of surjectivity.
Next Time

• **Graphs**
  • A ubiquitous, expressive, and flexible abstraction!

• **Properties of Graphs**
  • Building high-level structures out of lower-level ones!