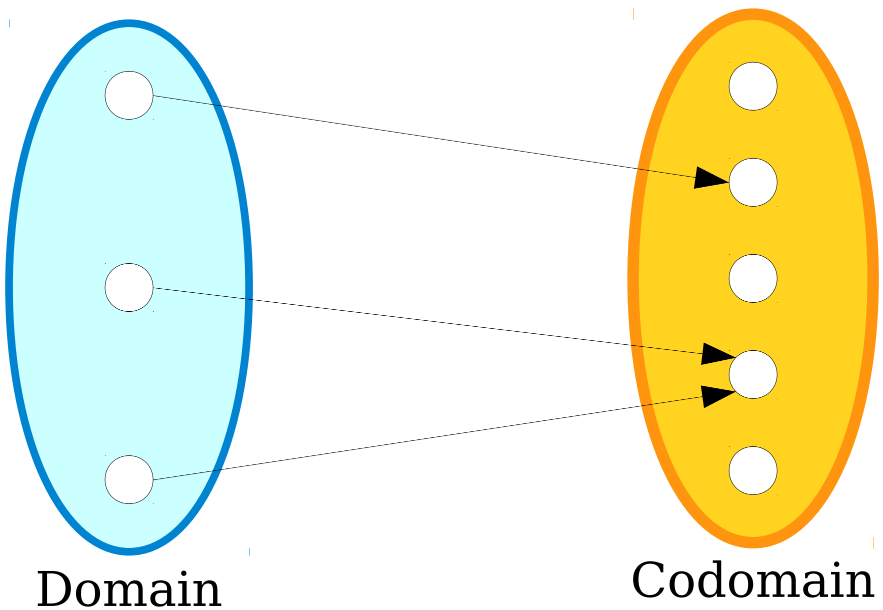
Cardinality

Recap from Last Time

Domains and Codomains

- Every function f has two sets associated with it: its ***domain*** and its ***codomain***.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.
- We write $\boldsymbol{f : A \to B}$ to indicate that f is a function whose domain is A and whose codomain is B.

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.

Function Composition

- If $f : A \to B$ and $g : B \to C$ are functions, the ***composition of f and g***, denoted $g \circ f$, is a function
  - whose domain is A,
  - whose codomain is C, and
  - which is evaluated as $(g \circ f)(x) = g(f(x))$.

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if each element of the codomain has at most one element of the domain that maps to it.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

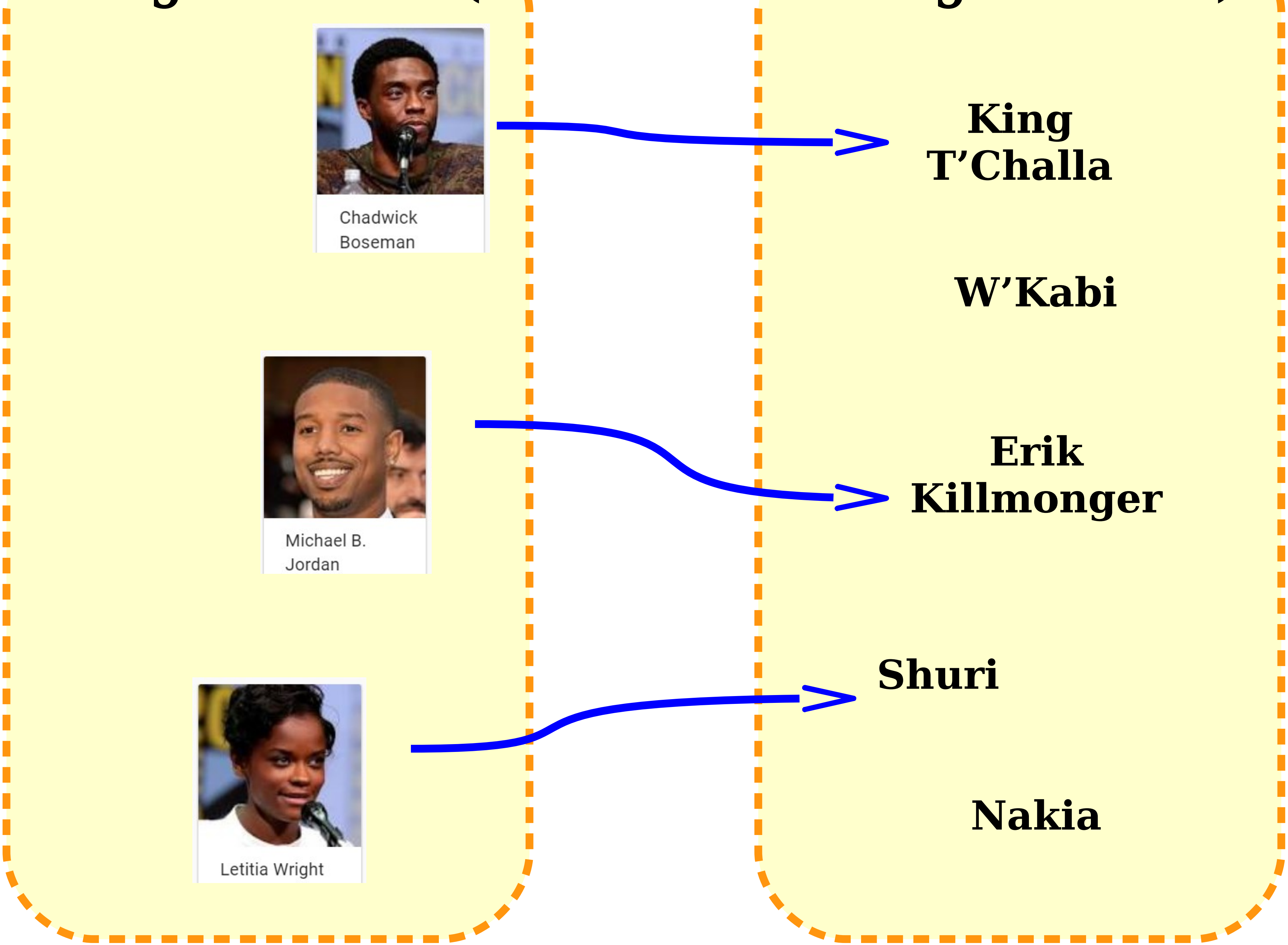
- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

- ***Theorem:*** The composition of two injections is an injection.

Injective (but not Surjective)



Surjective Functions

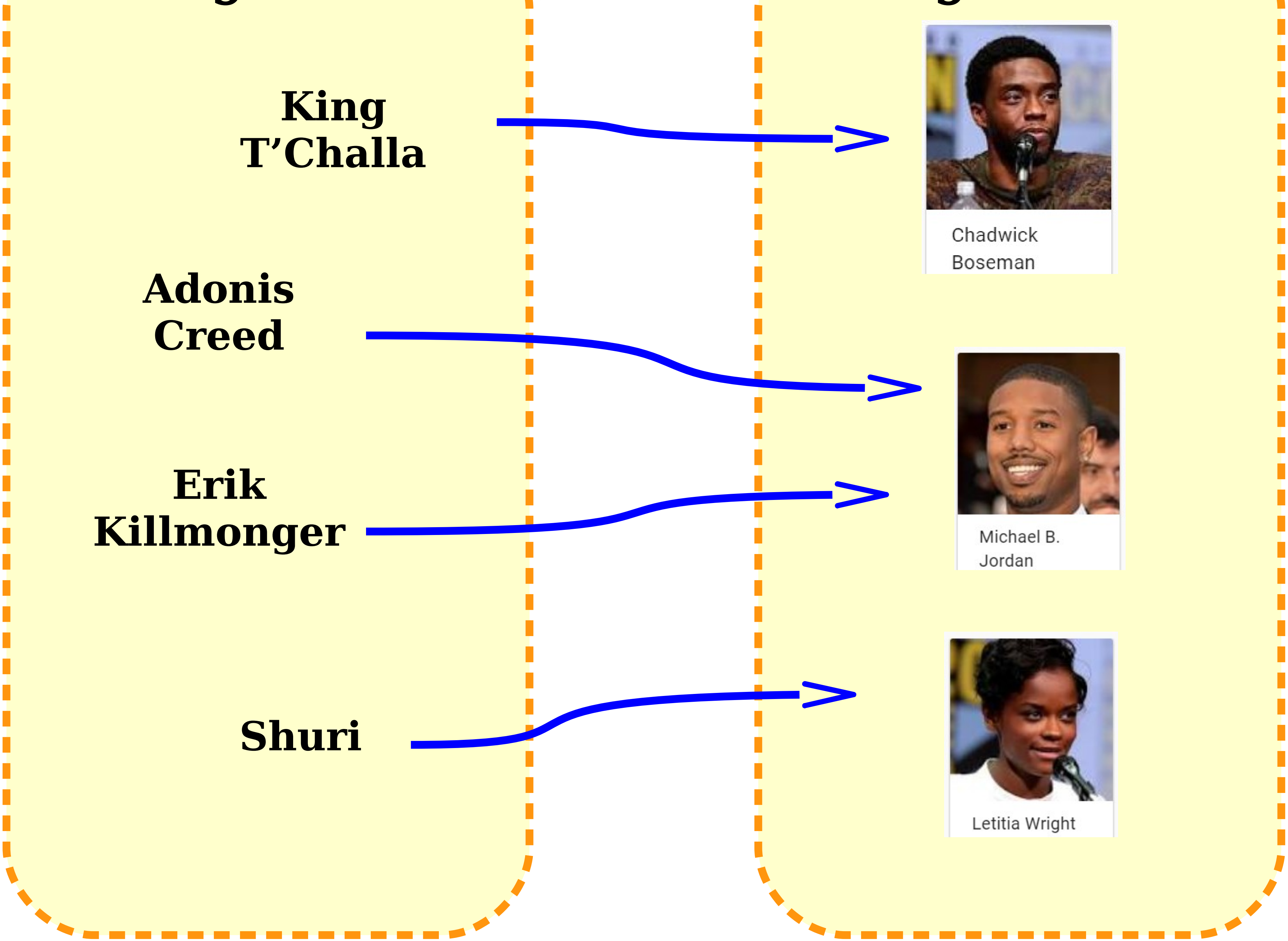
- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

- ***Theorem:*** The composition of two surjections is a surjection.

Surjective (but not Injective)

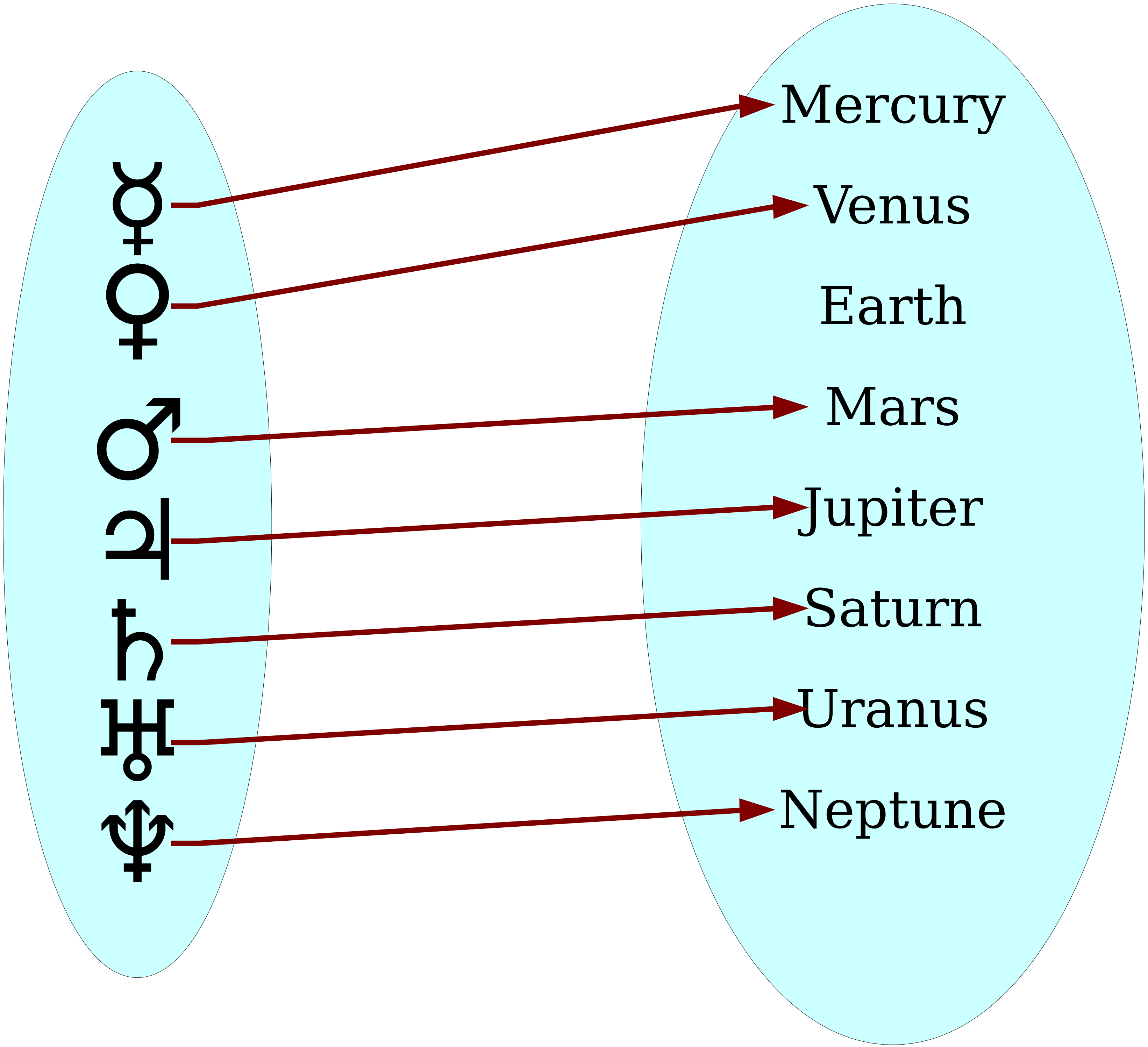


Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- ***Theorem:*** The composition of two bijections is a bijection.

New Stuff!

Inverse Functions



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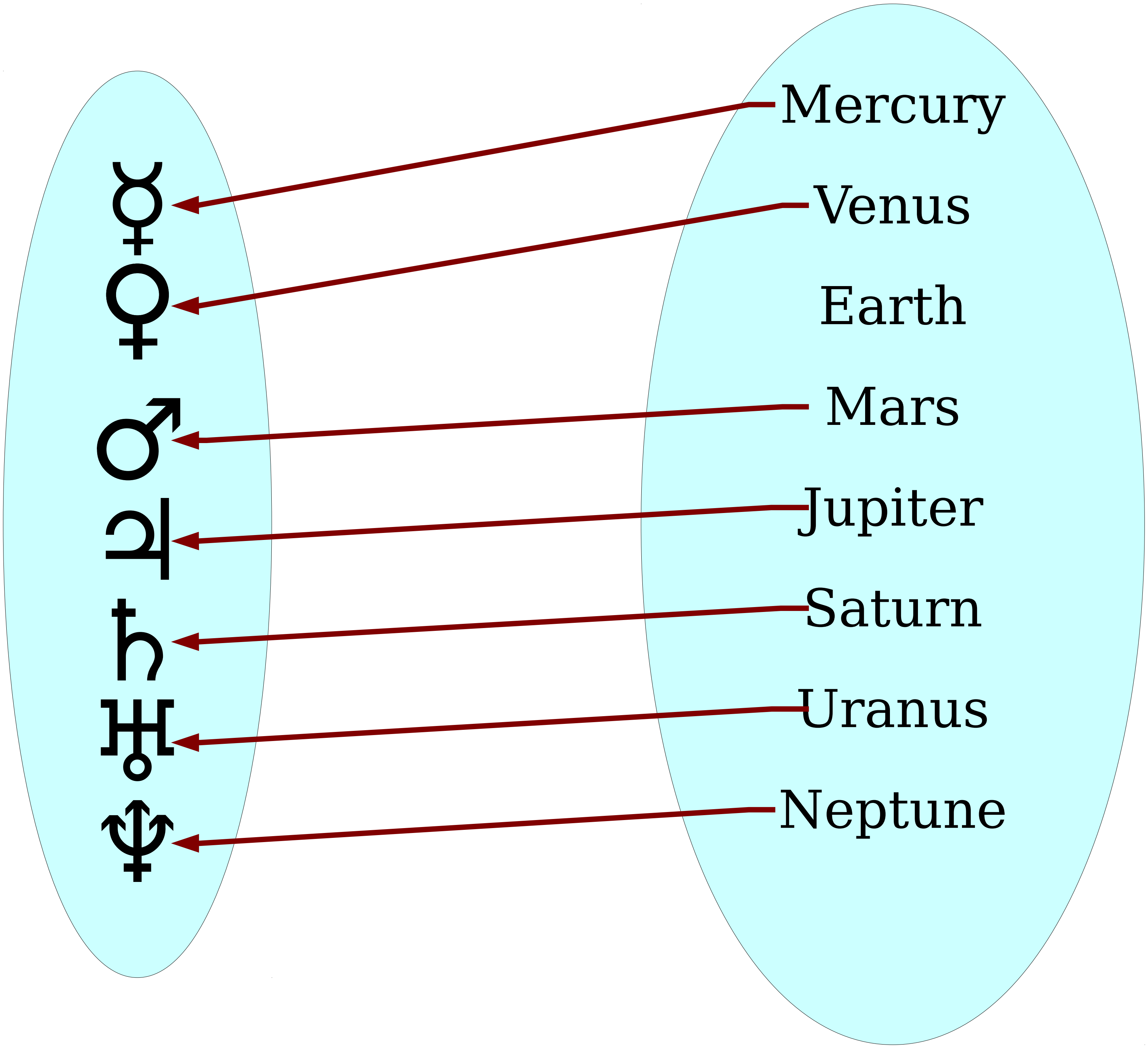

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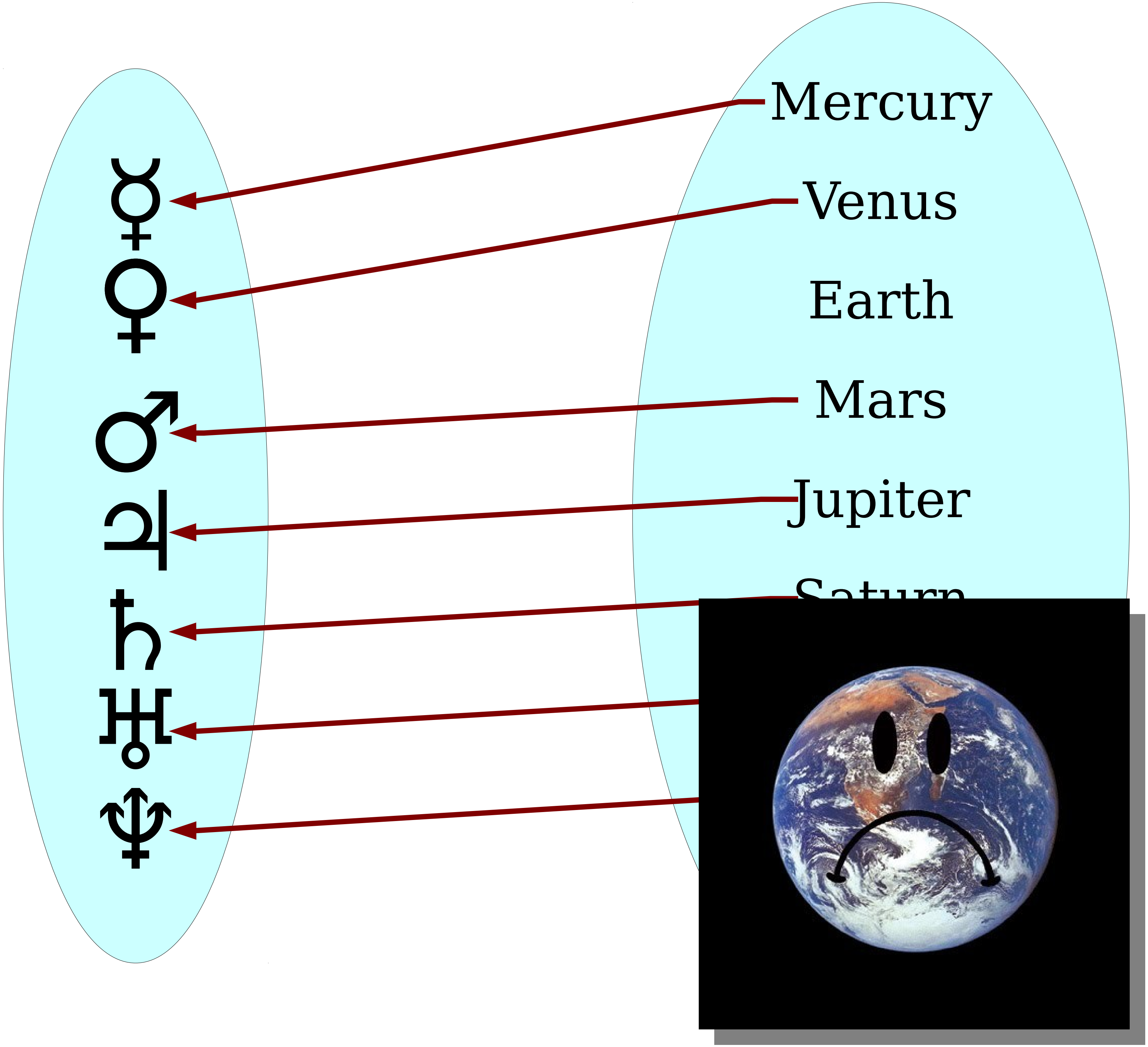
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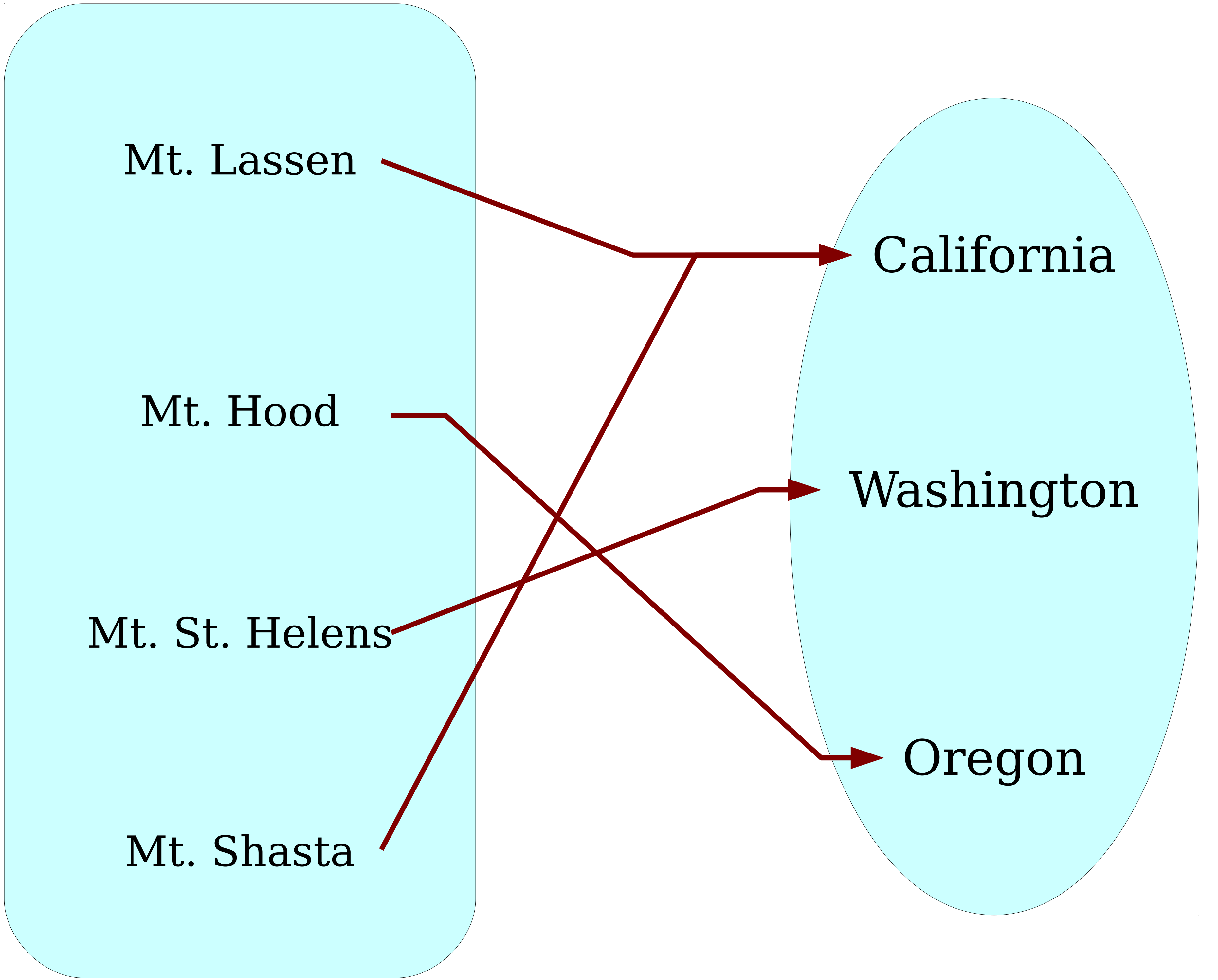
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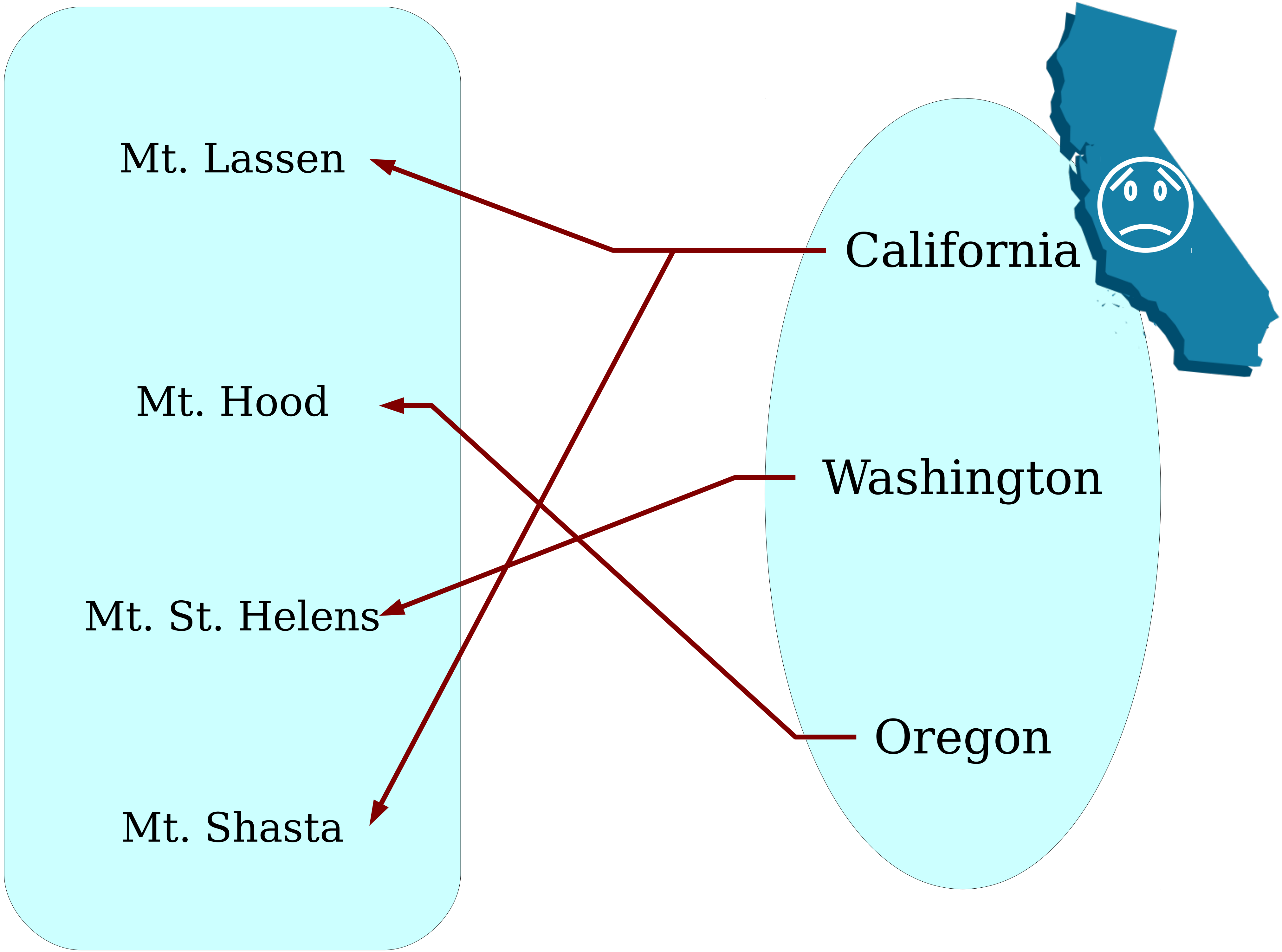
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California
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Inverse Functions

- In some cases, it's possible to "turn a function around."
- Let $f : A \to B$ be a function. A function $f^{-1} : B \to A$ is called an ***inverse of f*** if the following first-order logic statements are true about f and f^{-1}

$$\forall a \in A.\ (f^{-1}(f(a)) = a) \qquad \forall b \in B.\ (f(f^{-1}(b)) = b)$$

- In other words, if f maps a to b, then f^{-1} maps b back to a and vice-versa.
- Not all functions have inverses (we just saw a few examples of functions with no inverses).
- If f is a function that has an inverse, then we say that f is ***invertible***.

Inverse Functions

- ***Theorem:*** Let $f : A \to B$. Then f is invertible if and only if f is a bijection.
- This proof is in the course reader. Feel free to check it out if you'd like!

Where We Are

- We now know
  - what an injection, surjection, and bijection are;
  - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
  - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason…

Cardinality Revisited

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

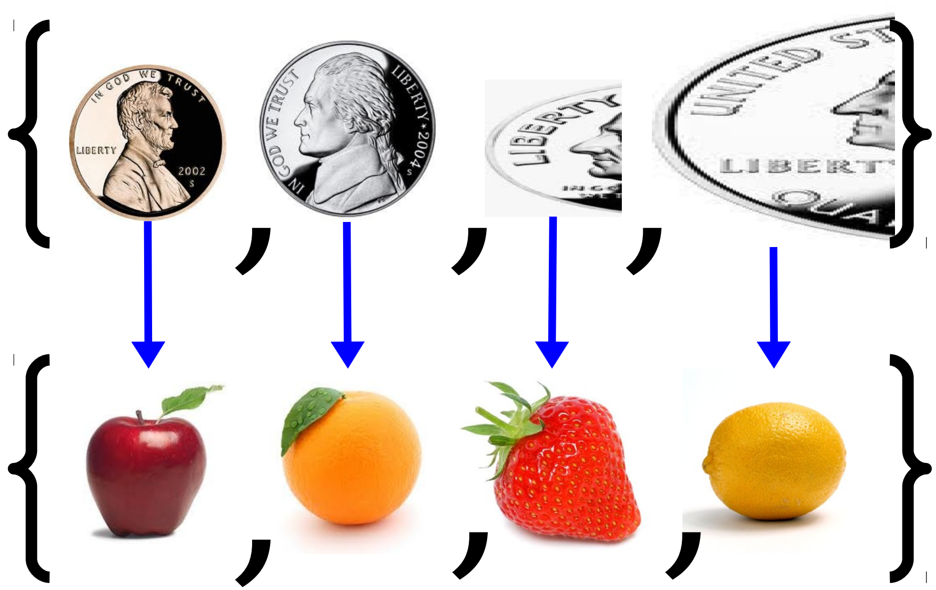
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
  - (Take Math 161 for an answer!)
- ***Idea:*** Define cardinality as a *relation* between two sets rather than an absolute quantity.

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$$|S| = |T| \text{ if there exists a } \textit{bijection } f : S \to T$$



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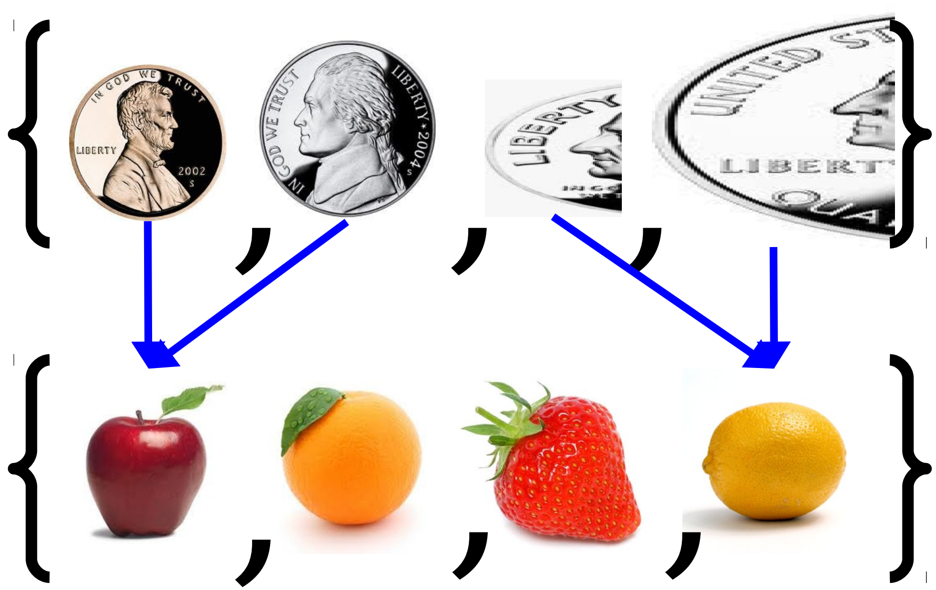

Fun with Cardinality

Terminology Refresher

- Let a and b be real numbers where $a \leq b$.
- The notation **$[a, b]$** denotes the set of all real numbers between a and b, inclusive.

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

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Home on the Range

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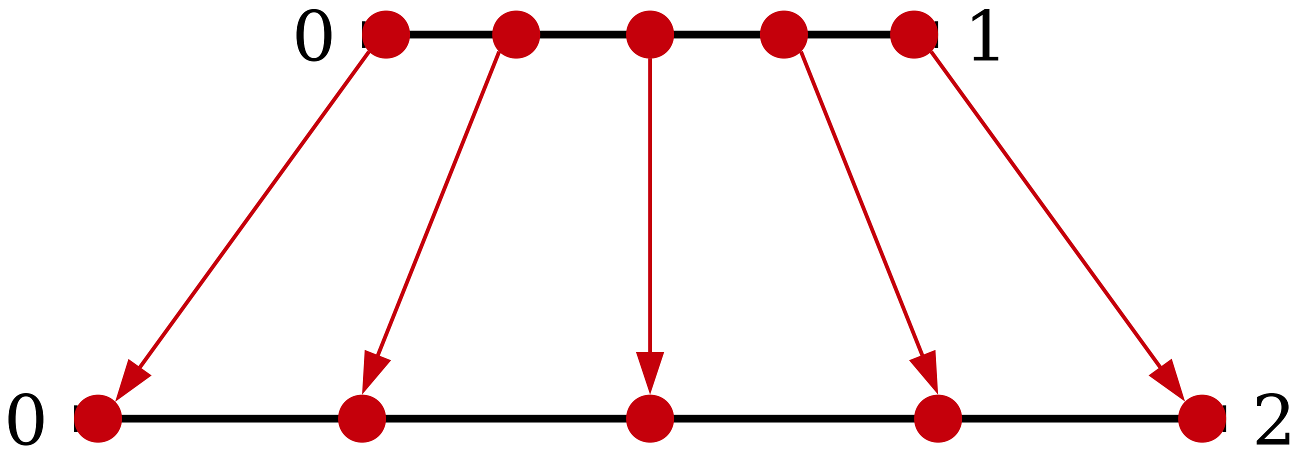

$$f : [0, 1] \to [0, 2]$$
$$f(x) = 2x$$

Theorem: $|[0, 1]| = |[0, 2]|$

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Proof:

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Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$.

How many of the following are proper ways of setting up the next part of this proof?

Choose any $x \in [0, 1]$. We will show there is a $y \in [0, 2]$ such that $f(x) = y$.

Pick any $y \in [0, 2]$. We will show there is an $x \in [0, 1]$ where $f(x) = y$.

Assume for the sake of contradiction that, for any $y \in [0, 2]$ and for any $x \in [0, 1]$, we have $f(x) \neq y$.

Finally, we will show that f is surjective.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then a number between **0** and **3**.

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Let $x = {}^{y}/_{2}$.

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Let $x = y/2$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq y/2 \leq 1$. We picked $x = y/2$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$.

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$$f(x) = 2x$$

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$$f(x) = 2x = 2({}^{y}/_{2})$$

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so $f(x) = y$, as required.

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Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that f is a bijection.

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Next, we'll show that f is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that f is surjective. To do so, consider any $y \in [0, 2]$. We'll show that there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = {}^{y}/_{2}$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq {}^{y}/_{2} \leq 1$. We picked $x = {}^{y}/_{2}$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 2x = 2({}^{y}/_{2}) = y,$$

so $f(x) = y$, as required. ■

Home on the Range

$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

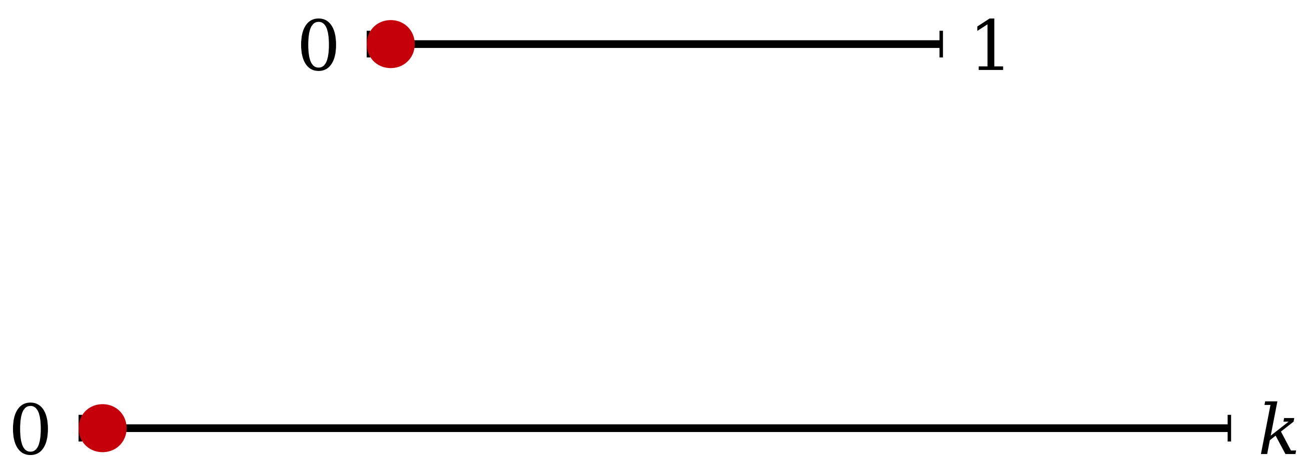
Home on the Range

$$f : [0, 1] \to [0, 3]$$
$$f(x) = 3x$$

Home on the Range

$$\boldsymbol{f : [0, 1] \to [0, 137]}$$
$$\boldsymbol{f(x) = 137x}$$

Home on the Range



(for any $k > 0$)

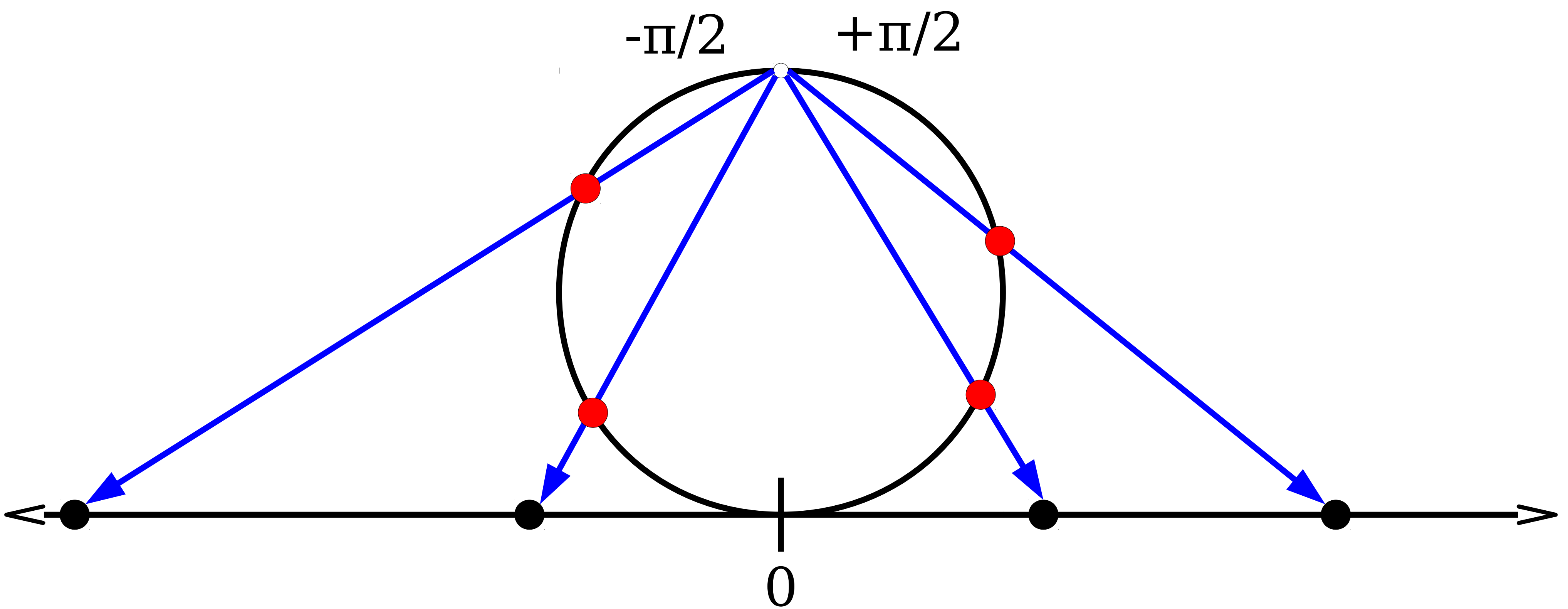
$$f : [0, 1] \to [0, k]$$

$$f(x) = kx$$

This means that ***cardinality*** (how many points there are) is a different idea than ***mass*** (how much those points weight). Look into ***measure theory*** if you're curious to learn more!

And one more example, just for funzies.

Put a Ring On It



$$f : (-\pi/2, \pi/2) \to \mathbb{R}$$

$$f(x) = \tan x$$

$$|(-\pi/2, \pi/2)| = |\mathbb{R}|$$

Some Properties of Cardinality

Theorem: For any set A, we have $|A| = |A|$.

Theorem: For any set A, we have $|A| = |A|$.

Proof:

Which of the following is the right high-level way to approach this proof?

A. Pick an arbitrary set A, then find a bijection $f : A \to A$.
B. Pick an arbitrary set A and show every function $f : A \to A$ is bijective.
C. There's nothing to prove here. Every object is equal to itself.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A***, ***B***, or ***C***.

Theorem: For any set A, we have $|A| = |A|$.

Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$.

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First, we'll show that f is a well-defined function.

Theorem: For any set A, we have $|A| = |A|$.

Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Theorem: For any set A, we have $|A| = |A|$.

Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective.

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First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$.

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First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$.

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First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

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Finally, we'll show that f is surjective.

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Finally, we'll show that f is surjective. Consider any $y \in A$.

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Finally, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$.

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Finally, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required.

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Theorem: If A, B, and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$.

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Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

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Consider the function $g \circ f : A \to C$.

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Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

Consider the function $g \circ f : A \to C$. Since g and f are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from A to C.

Theorem: If A, B, and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Great exercise: Prove that if A and B are sets where $|A| = |B|$, then $|B| = |A|$.

Time-Out for Announcements!

Midterm Exam Logistics

- Our first midterm exam is next ***Monday, April 30th***, from ***7:00PM – 10:00PM*** in **Cubberley** Auditorium.
  - You're responsible for Lectures 00 – 05 and topics covered in PS1 – PS2. Later lectures (relations forward) and problem sets (PS3 onward) won't be tested here.
  - The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam.
  - Students with OAE accommodations: we will be reaching out to you soon with room and time assignments.

Practice Midterm Exam

- To help you prepare for the midterm, we'll be holding a practice midterm exam ***tonight*** from ***7PM – 10PM*** in ***Cemex Auditorium***.
  - The exam we’ll use isn’t one of the ones posted up on the course website, so feel free to use those as practice in the meantime.
- The practice midterm exam is an actual midterm we gave out in a previous quarter. It’s probably the best indicator of what you should expect to see.
- Course staff will be on hand to answer your questions.
- Can't make it? We'll release that practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!

Back to CS103!

Unequal Cardinalities

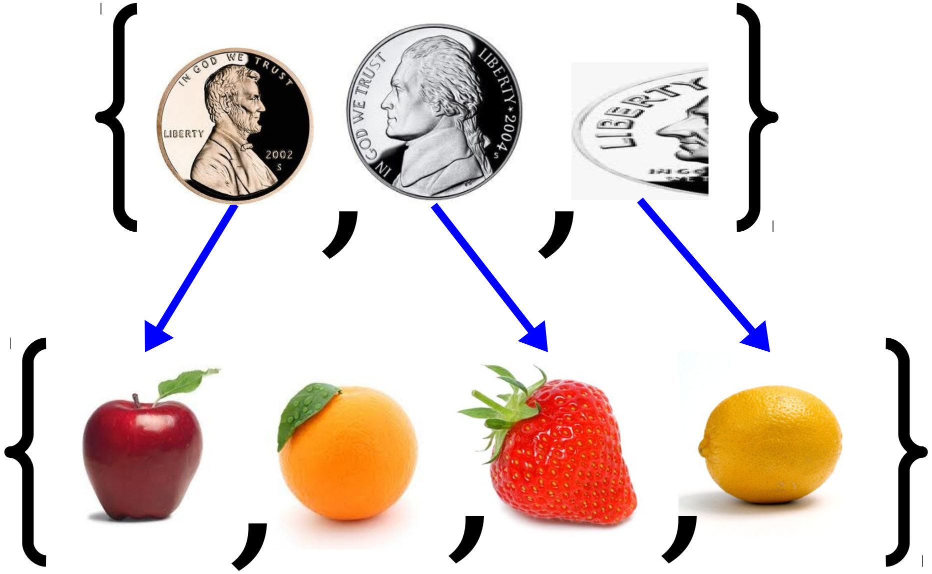
- Recall: $|A| = |B|$ if the following statement is true:

There exists a bijection $f : A \to B$

- What does it mean for $|A| \neq |B|$ to be true?

Every function $f : A \to B$ is not a bijection.

- This is a strong statement! To prove $|A| \neq |B|$, we need to show that *no possible function* from A to B can be injective and surjective.



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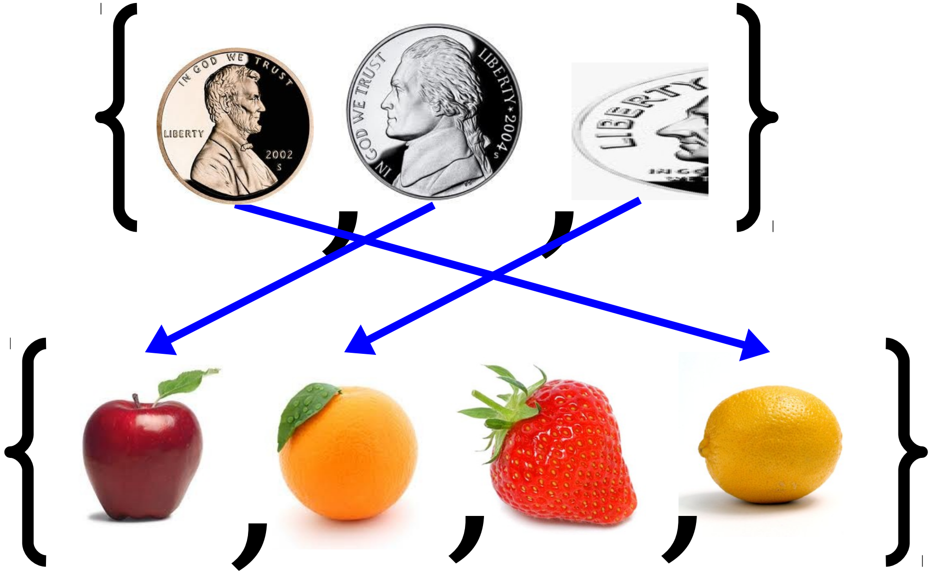

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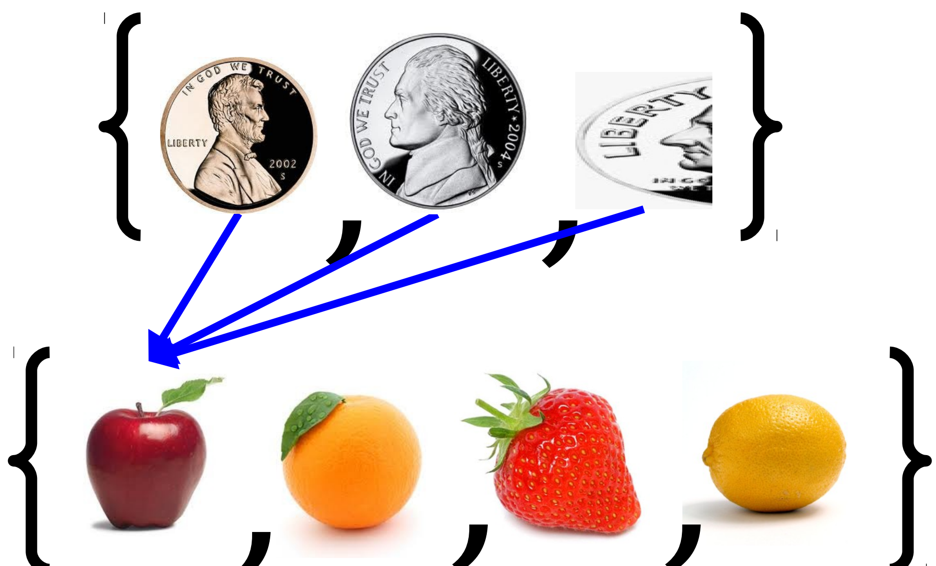

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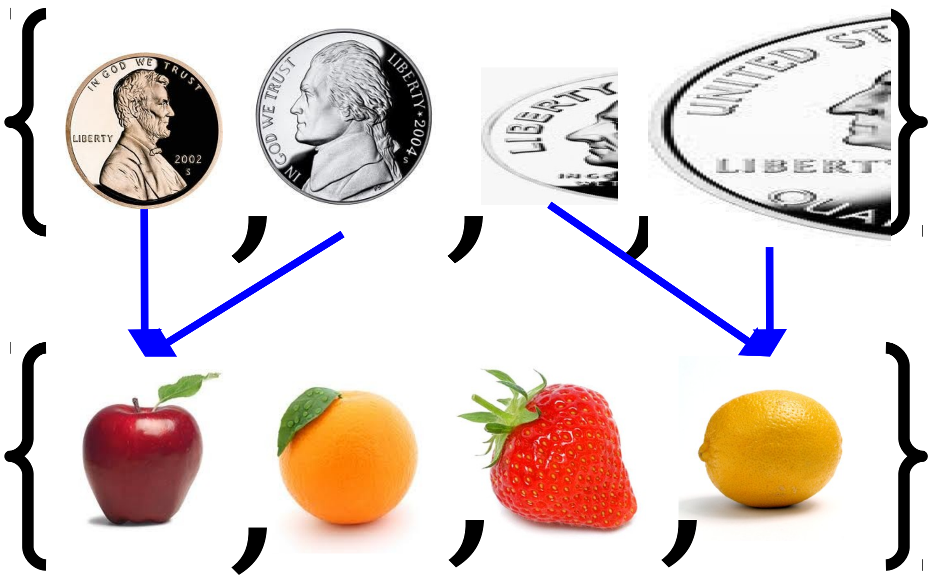

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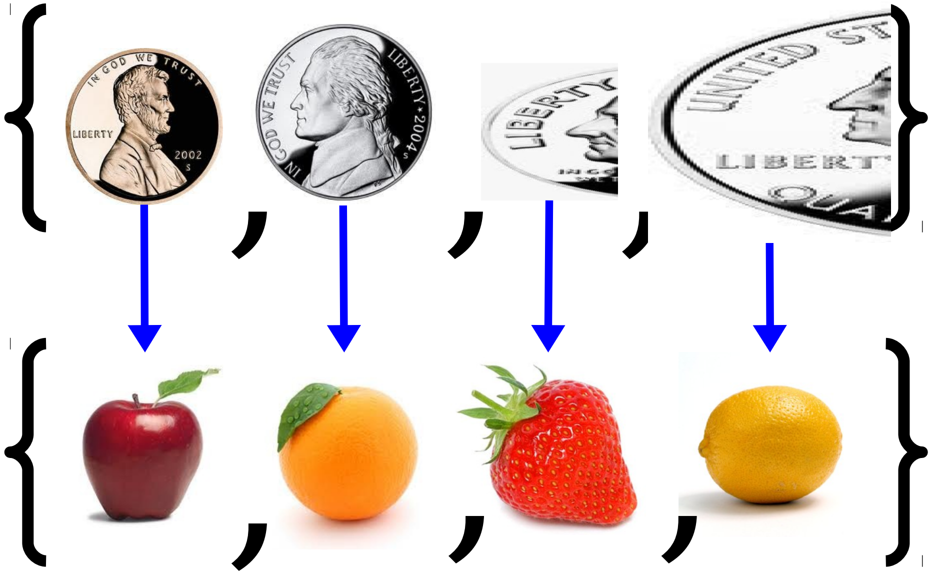

Cantor’s Theorem Revisited

Cantor's Theorem

- In our very first lecture, we sketched out a proof of ***Cantor's theorem***, which says that

If S is a set, then $|S| < |\wp(S)|$.

- That proof was visual and pretty hand-wavy. Let's see if we can go back and formalize it!

Where We're Going

- Today, we're going to formally prove the following result:

If S is a set, then $|S| \neq |\wp(S)|$.

- We've released an online Guide to Cantor's Theorem, which will go into *way* more depth than what we're going to see here.
- The goal for today will be to see how to start with our picture and turn it into something rigorous.
- On the next problem set, you'll explore the proof in more depth and see some other applications.

The Roadmap

- We're going to prove this statement:

$$\text{If } S \text{ is a set, then } |S| \neq |\wp(S)|.$$

- Here's how this will work:
  - Pick an arbitrary set S.
  - Pick an arbitrary function $f : S \to \wp(S)$.
  - Show that f is not surjective using a diagonal argument.
  - Conclude that there are no bijections from S to $\wp(S)$.
  - Conclude that $|S| \neq |\wp(S)|$.

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- Pick an arbitrary function $f : S \to \wp(S)$.
- Show that f is not surjective using a diagonal argument.
- Conclude that there are no bijections from S to $\wp(S)$.
- Conclude that $|S| \neq |\wp(S)|$.

x_0

x_1

x_2

x_3

x_4

x_5

$\dots$

This is a drawing of our function $f : S \to \wp(S)$.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \ldots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \ldots \}$$

$$x_2 \longrightarrow \{ x_4, \ldots \}$$

$$x_3 \longrightarrow \{ x_1, x_2, x_4, \ldots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$$

$$\ldots$$

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x_0	x_1	x_2	x_3	x_4	x_5	…

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…

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	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**

$x_1 \to \{ x_0, x_3, x_4, \ldots \}$

$x_2 \to \{ x_4, \ldots \}$

$x_3 \to \{ x_1, x_2, x_4, \ldots \}$

$x_4 \to \{ x_0, x_5, \ldots \}$

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…

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	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \to$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \to \{ x_4, \dots \}$

$x_3 \to \{ x_1, x_2, x_4, \dots \}$

$x_4 \to \{ x_0, x_5, \dots \}$

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…

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	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \to$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \to \{ \quad x_4, \quad \dots \}$

$x_3 \to \{ x_1, x_2, x_4, \dots \}$

$x_4 \to \{ x_0, x_5, \dots \}$

$x_5 \to \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

$\dots$

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \to$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \to$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \to \{ x_1, x_2, x_4, \dots \}$

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> *This is a drawing of our function $f : S \to \wp(S)$.*

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \to$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \to$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \to \{ x_1, x_2, x_4, \dots \}$

$x_4 \to \{ x_0, x_5, \dots \}$

$x_5 \to \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \to$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \to$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \to$	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**

$x_4 \to \{ x_0, x_5, \ldots \}$

$x_5 \to \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 →	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1 →	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2 →	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3 →	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**
x_4 →	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**

$x_5 \to \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…							

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$\{\ \boldsymbol{x_0},\ \boldsymbol{x_5},\ \dots\ \}$

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
					Y	**N**	…
					N	**Y**	…
…	…	…	…	…	…	…	…

For this choice of f, what set is shown in blue below?

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then your answer.

Flip all Y's to N's and vice-versa to get a new set

N	**Y**	**Y**	**Y**	**Y**	**N**	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Flip all Y's to N's and vice-versa to get a new set

$$\{ \quad x_1, \; x_2, \; x_3, \; x_4, \quad \dots \}$$

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

What set is this?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

N	**Y**	**Y**	**Y**	**Y**	**N**	...

What set is this?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

N	**Y**	**Y**	**Y**	**Y**	**N**	...

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

$f(x_0)$

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$x_0 \in f(x_0)$?

N	**Y**	**Y**	**Y**	**Y**	**N**	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

$\boldsymbol{x_0 \notin f(x_0)?}$

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$f(x_1)$

N	**Y**	**Y**	**Y**	**Y**	**N**	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

$x_1 \in f(x_1)$?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	$\boxed{\textbf{Y}}$	**Y**	**Y**	**Y**	**N**	…

$x_1 \notin f(x_1)$?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$f(x_2)$

N	**Y**	**Y**	**Y**	**Y**	**N**	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

$\boldsymbol{x_2 \in f(x_2)?}$

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	$\boxed{\textbf{Y}}$	**Y**	**Y**	**N**	…

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x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$x_3 \notin f(x_3)$?

N	**Y**	**Y**	$\boxed{\textbf{Y}}$	**Y**	**N**	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$x_4 \notin f(x_4)$?

N	**Y**	**Y**	**Y**	$\boxed{\textbf{Y}}$	**N**	…

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	$\boxed{\textbf{N}}$	…

$x_5 \notin f(x_5)$?

This is a drawing of our function $f : S \to \wp(S)$.

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$x \notin f(x)$?

N	**Y**	**Y**	**Y**	**Y**	**Y**	**N**	…

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	x_0	x_1	x_2	x_3	x_4	x_5	…
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x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**Y**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$\{\ x \in S \mid x \notin f(x)\ \}$

N	**Y**	**Y**	**Y**	**Y**	**N**	…

The Diagonal Set

- For any set S and function $f : S \to \wp(S)$, we can define a set D as follows:

$$\boldsymbol{D = \{ x \in S \mid x \notin f(x) \}}$$

("The set of all elements x where x is not an element of the set f(x).")

- This is a formalization of the set we found in the previous picture.
- Using this choice of D, we can formally prove that no function $f : S \to \wp(S)$ is a bijection.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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Starting with f, we define the set

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The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor's theorem can be formalized in terms of surjectivity.

Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!