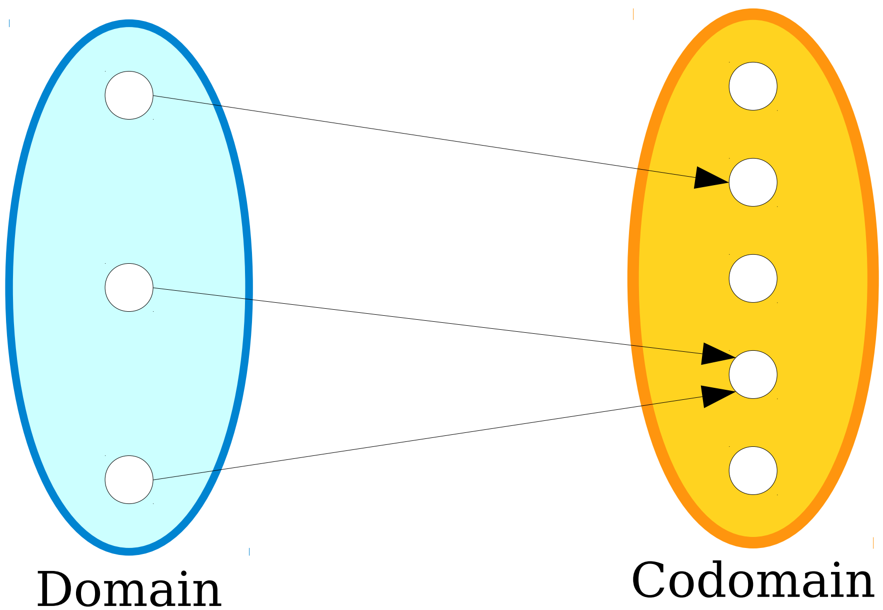
# Cardinality

# Recap from Last Time

# Domains and Codomains

- Every function $f$ has two sets associated with it: its ***domain*** and its ***codomain***.
- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.
- We write $f : A \to B$ to indicate that $f$ is a function whose domain is $A$ and whose codomain is $B$.

The function must be defined for each element of its domain.




The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.

# Function Composition

- Let $f : A \to B$ and $g : B \to C$ be arbitrary functions.
- The ***composition of f and g***, denoted $g \circ f$, is a function
  - whose domain is $A$,
  - whose codomain is $C$, and
  - which is evaluated as $(g \circ f)(x) = g(f(x))$.

# Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if each element of the codomain has at most one element of the domain that maps to it.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

- ***Theorem:*** The composition of two injections is an injection.

# Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

- ***Theorem:*** The composition of two surjections is a surjection.

# Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- ***Theorem:*** The composition of two bijections is a bijection.

# Where We Are

- We now know
  - what an injection, surjection, and bijection are;
  - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
  - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason…

# New Stuff!

# Cardinality Revisited

# Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If $S$ is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

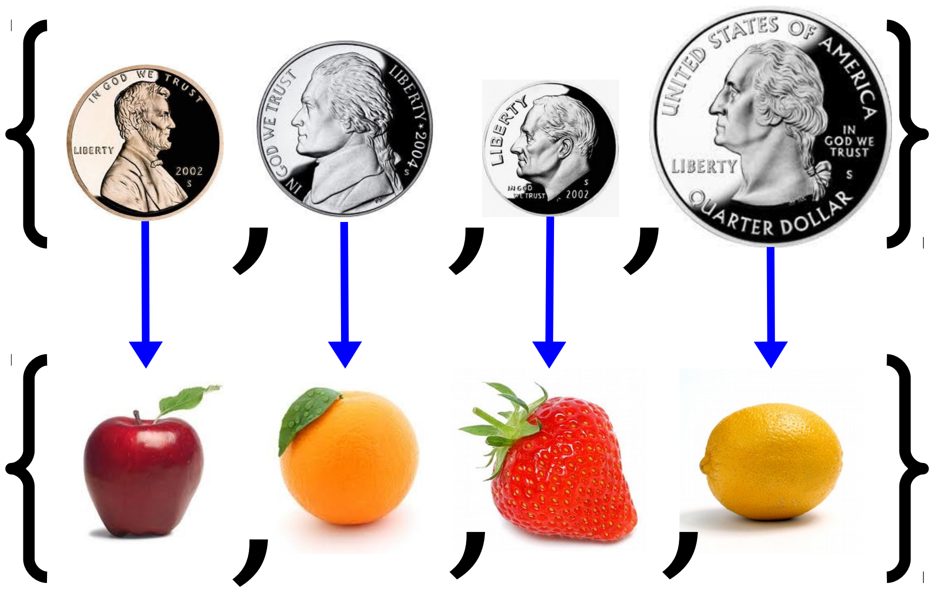
# Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
  - (Take Math 161 for an answer!)
- ***Idea:*** Define cardinality as a *relation* between two sets rather than an absolute quantity.

# Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$$|S| = |T| \textbf{ if there exists a } \textbf{\textit{bijection}}\ f : S \to T$$



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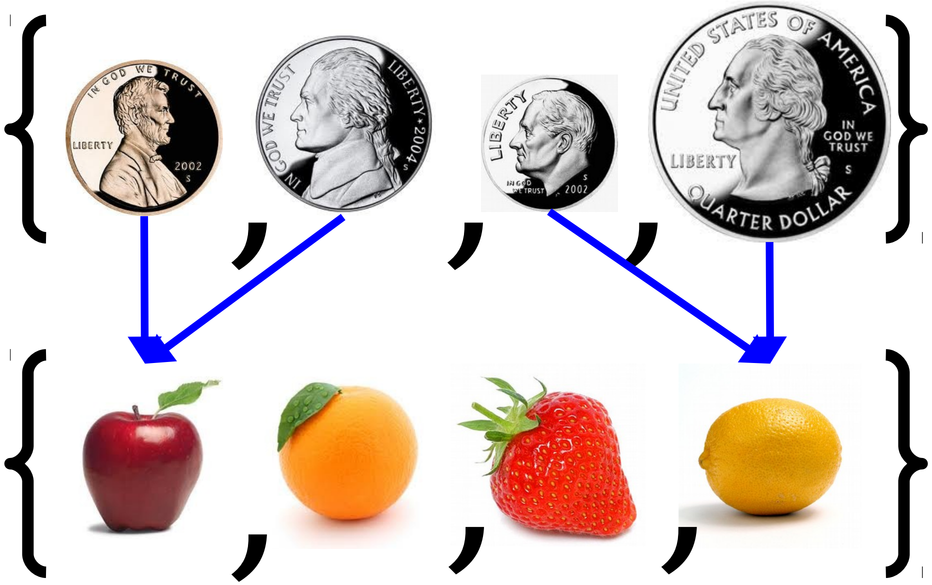

# Properties of Cardinality

- For any sets $A$, $B$, and $C$, the following are true:
  - **$|A| = |A|$.**
    - Define $f : A \to A$ as $f(x) = x$.
  - **If $|A| = |B|$, then $|B| = |A|$.**
    - If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
  - **If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.**
    - If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

# Fun with Cardinality

# Infinite Cardinalities

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{N}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … |
| $\mathbb{Z}$ | … | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | … |

# Infinite Cardinalities

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|---|---|---|---|---|---|---|---|---|---|---|
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| | $\downarrow$ | | $\downarrow$ | | $\downarrow$ | | $\downarrow$ | | $\downarrow$ | |
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# Infinite Cardinalities

$$
\begin{tikzcd}[column sep=small]
\mathbb{N} & 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & 4 \arrow[d] & 5 \arrow[d] & 6 \arrow[d] & 7 \arrow[d] & 8 \arrow[d] & \ldots \\
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\end{tikzcd}
$$

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Define $f : \mathbb{N} \to \mathbb{Z}$ as follows:

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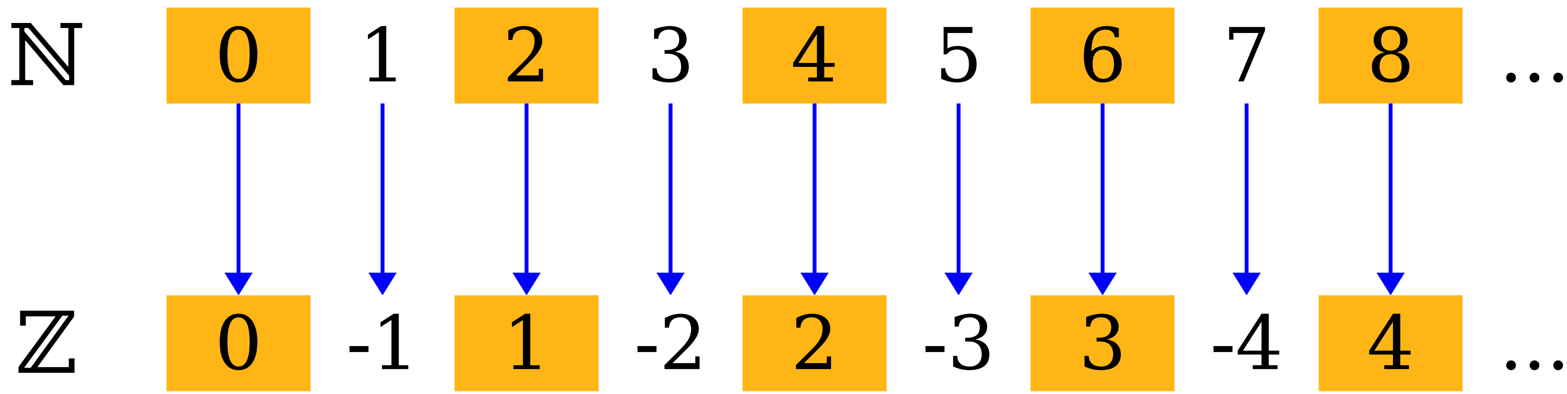
$$f(n) = \left\{ \right.$$

# Infinite Cardinalities




Define $f : \mathbb{N} \to \mathbb{Z}$ as follows:

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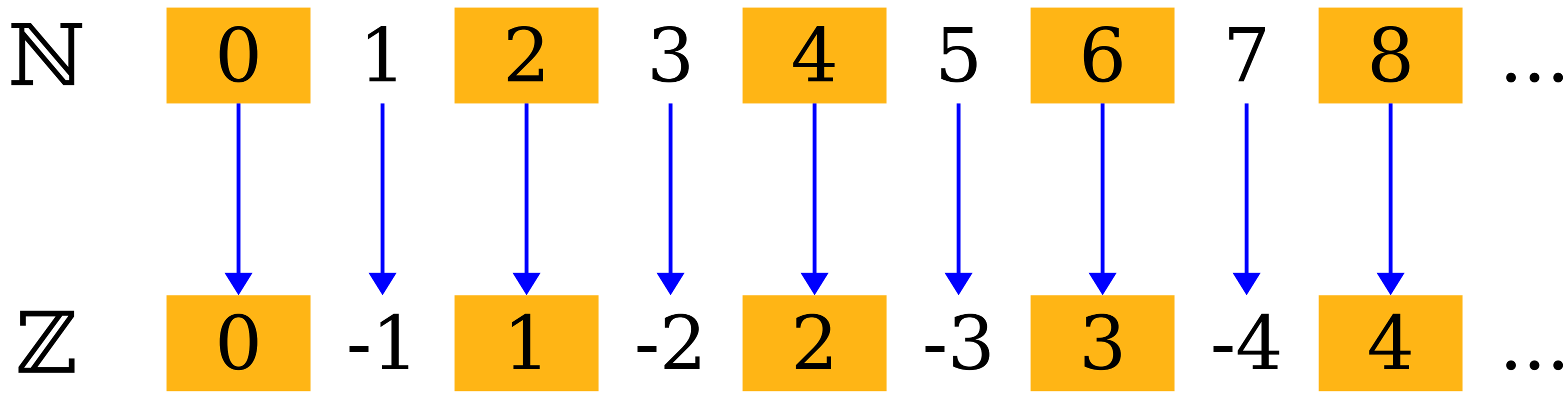


Define $f : \mathbb{N} \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ & \end{cases}$$

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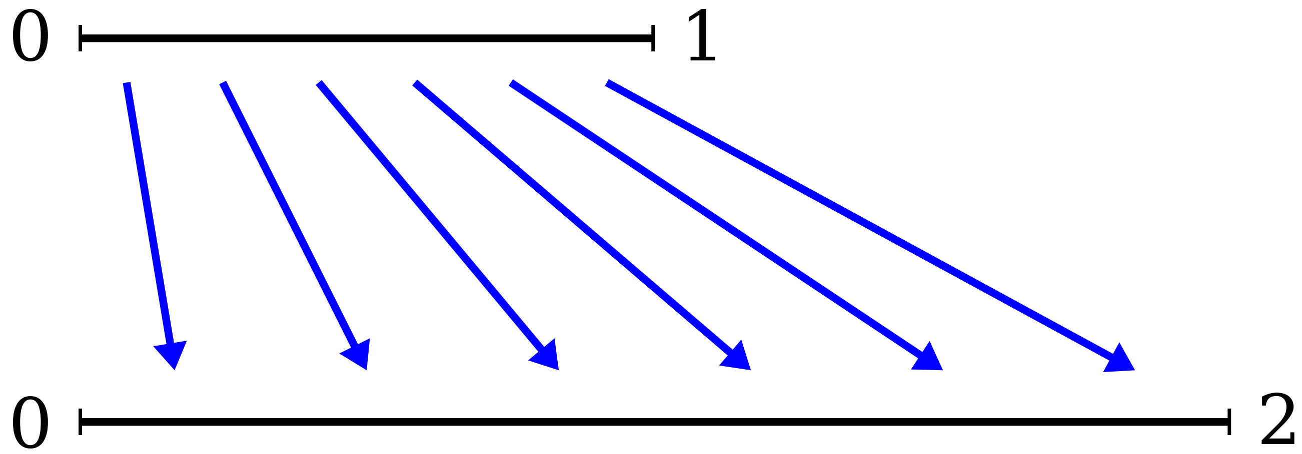
Define $f : \mathbb{N} \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{otherwise} \end{cases}$$

# Home on the Range

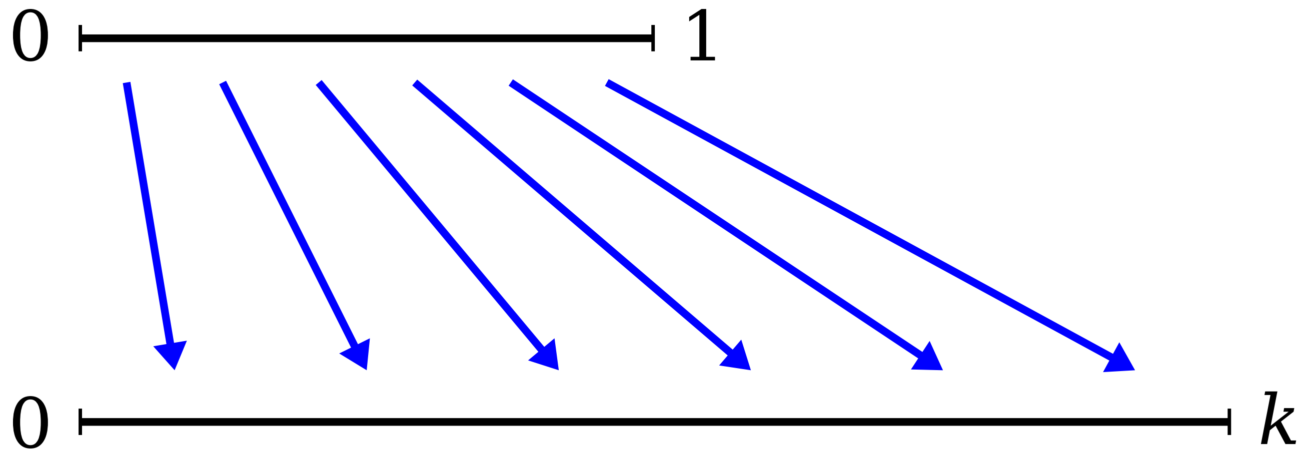



$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

$$|[0, 1]| = |[0, 2]|$$

# Home on the Range



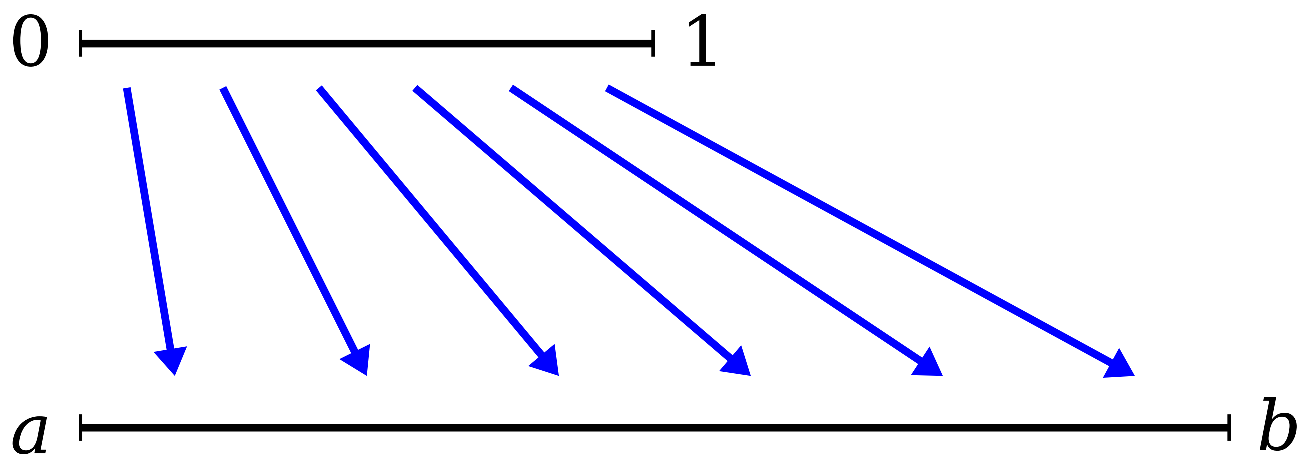

For any $k > 0$:

$$\boldsymbol{f : [0, 1] \to [0, k]}$$

$$\boldsymbol{f(x) = kx}$$

$$\boldsymbol{|[0, 1]| = |[0, k]|}$$

# Home on the Range



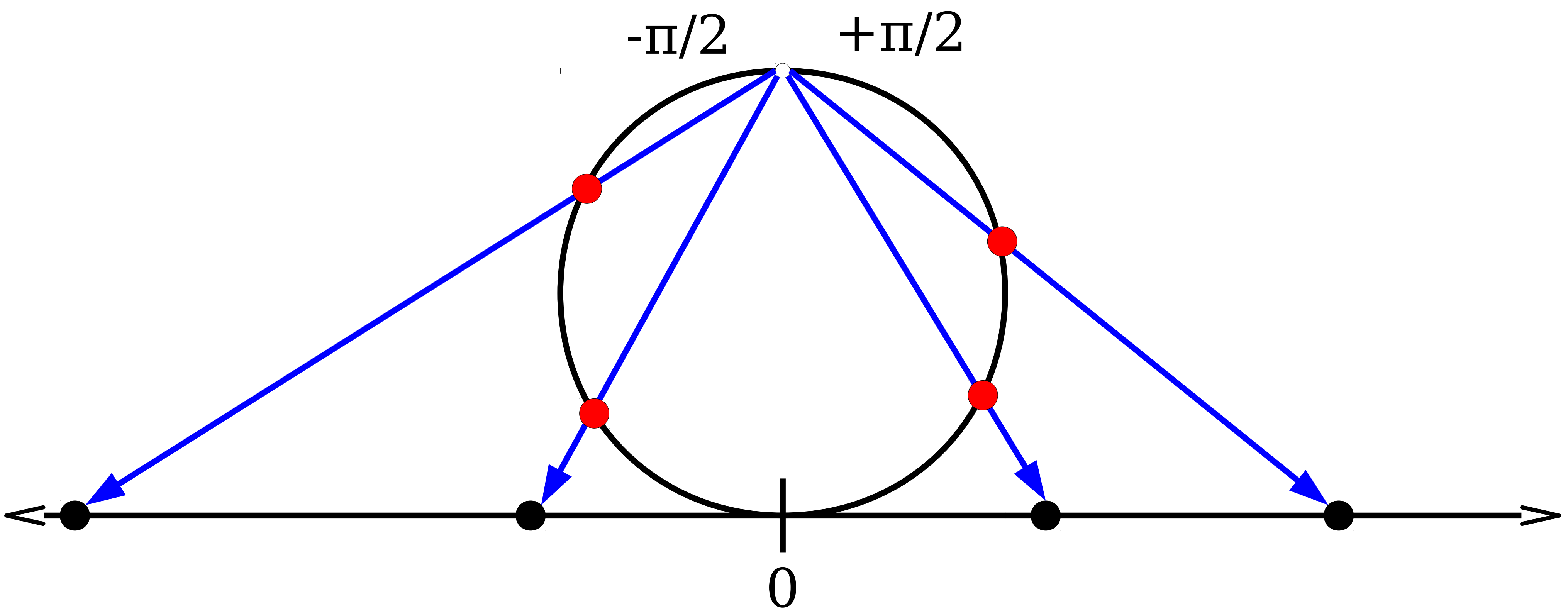

For any $a < b$:

$\boldsymbol{f : [0, 1] \to [a, b]}$

$\boldsymbol{f(x) = (b - a)x + a}$

$\boldsymbol{|[0, 1]| = |[a, b]|}$

# Put a Ring On It




$f : (-\pi/2, \pi/2) \to \mathbb{R}$

$f(x) = \tan x$

$|(-\pi/2, \pi/2)| = |\mathbb{R}|$

# The Cartesian Product

- The ***Cartesian product*** of $A \times B$ of two sets is defined as

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| | **a** | **b** | **c** |
|---|---|---|---|
| **0** | | | |
| **1** | | | |
| **2** | | | |

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| | ***a*** | ***b*** | ***c*** |
|---|---|---|---|
| **0** | **(0, *a*)** | **(0, *b*)** | **(0, *c*)** |
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$$\underbrace{\left\{ 0, 1, 2 \right\}}_{A} \times \underbrace{\left\{ a, b, c \right\}}_{B} = \left\{ \begin{array}{l} (0, a), (0, b), (0, c), \\ (1, a), (1, b), (1, c), \\ (2, a), (2, b), (2, c) \end{array} \right\}$$

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$$\underbrace{\{0, 1, 2\}}_{A} \times \underbrace{\{0, 1, 2\}}_{A} = \left\{ \begin{array}{l} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{array} \right\}$$

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$$\underbrace{\{0, 1, 2\}^2}_{A^2} = \left\{ \begin{array}{l} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{array} \right\}$$

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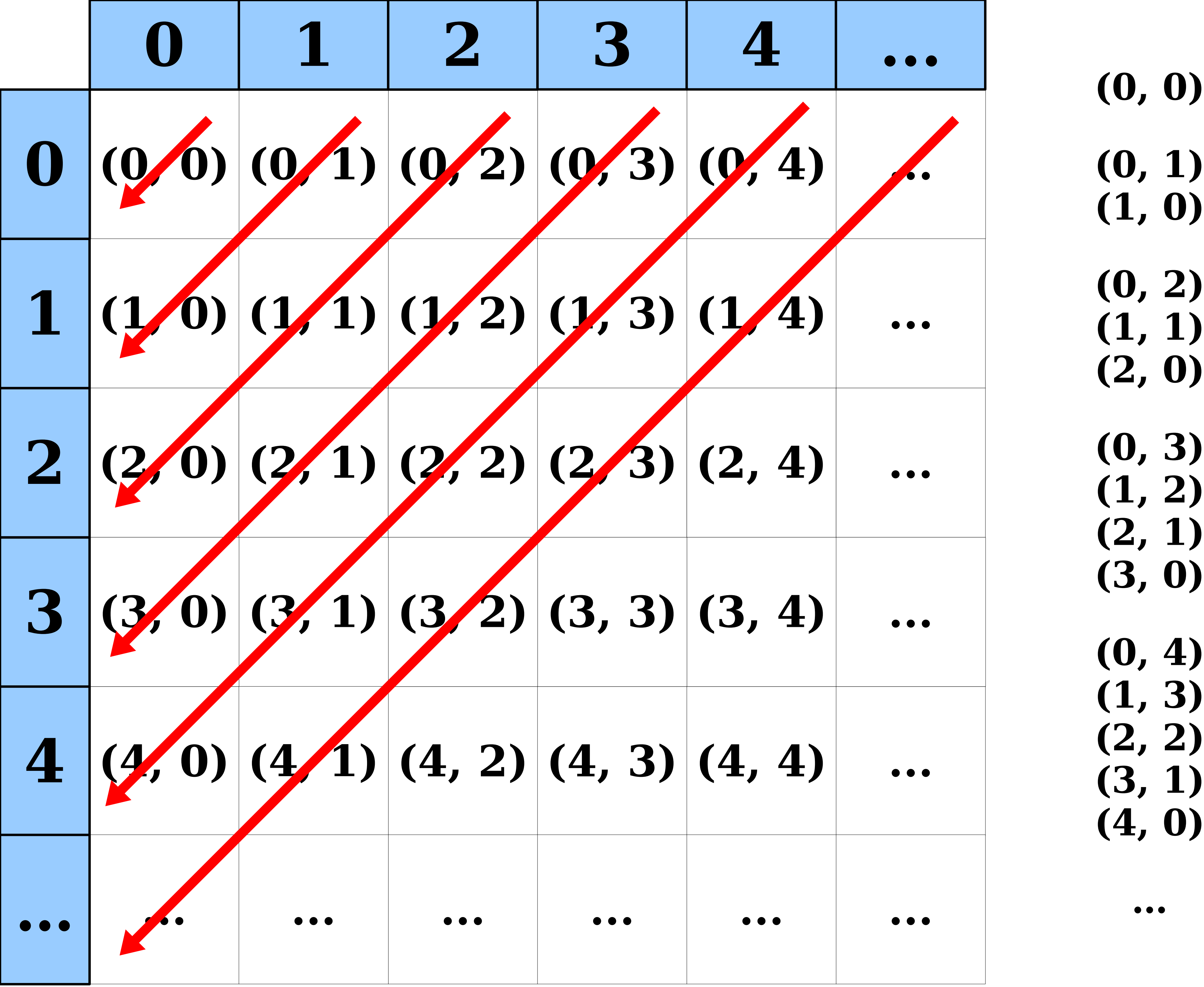
If you've taken Math 51, this is where we get the notation $\mathbb{R}^2$ and $\mathbb{R}^3$ from!

$$\underbrace{\left\{ 0, 1, 2 \right\}^2}_{A^2} = \left\{ \begin{array}{l} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{array} \right\}$$

What is $|\mathbb{N}^2|$?



0 1 2 3 4 ...
0 (0, 0) (0, 1) (0, 2) (0, 3) (0, 4) ...
1 (1, 0) (1, 1) (1, 2) (1, 3) (1, 4) ...
2 (2, 0) (2, 1) (2, 2) (2, 3) (2, 4) ...
3 (3, 0) (3, 1) (3, 2) (3, 3) (3, 4) ...
4 (4, 0) (4, 1) (4, 2) (4, 3) (4, 4) ...
... ... ... ... ... ... ...
(0, 0)
(0, 1)
(1, 0)
(0, 2)
(1, 1)
(2, 0)
(0, 3)
(1, 2)
(2, 1)
(3, 0)
(0, 4)
(1, 3)
(2, 2)
(3, 1)
(4, 0)
...

Counterintuitive result: $|\mathbb{N}| = |\mathbb{N}^2|$

Time-Out for Announcements!

# WiCS Casual CS Dinner

- WiCS is holding a Casual CS Dinner tonight from 6:00PM – 7:00PM in the WCC.
- Show up, meet cool folks, get some dinner, and have fun!
- Everyone is welcome!

# Apply to Section Lead!

- Like CS? Want to contribute back to the community? Want to get folks excited about the field? Want to meet cool people and get a steady job on campus? Want to get into a nifty professional network?

***Apply to section lead!***

- Deadline is
  - this Thursday if you've completed CS106B/X, and
  - Thursday, May 11 if you are currently in CS106B/X.

# Problem Sets

- The PS3 checkpoint was due at the start of class today. We'll get it graded and returned to you by Wednesday.
- PS3 is due this Friday. You *can* use late days on it, but it might not be a good idea to do that given that the midterm is next Tuesday.
- PS2 solutions are now available! You should definitely read over them and make sure you understand the answers. We'll get graded PS2's back to you by Wednesday as well.

# Midterm Exam Logistics

- The first midterm exam is next ***Tuesday, May 2nd***, from ***7:00PM – 10:00PM***. Locations are divvied up by last (family) name:
  - `Abb – Niu`: Go to Hewlett 200.
  - `Nor – Vas`: Go to Hewlett 201.
  - `Vil – Yim`: Go to Hewlett 102.
  - `You – Zuc`: Go to Hewlett 103.
- You're responsible for Lectures 00 – 05 and topics covered in PS1 – PS2. Later lectures and problem sets won't be tested.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.
- Students with OAE accommodations: please contact us ***immediately*** if you haven't yet done so. We'll ping you about setting up alternate exams.

# Midterm Exam

- ***We want you to do well on this exam.*** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your "mathematical potential" or "innate mathematical ability."

# Practice Midterm Exam

- To help you prepare for the midterm, we'll be holding a practice midterm exam tomorrow, ***Tuesday, April 25***, from ***7PM – 10PM***, tentatively scheduled in Dink.
- The practice midterm exam is composed of previous exam questions used in CS103. It's similar in form and style to the upcoming midterm.
- Course staff will be on hand to answer your questions.
- Can't make it? We'll release the practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!

# Extra Practice Problems

- We'll be releasing three sets of cumulative review problems this week that you can use to prepare for the exam.
- We strongly recommend working through these practice problems. They're a great way to get additional practice with the material and to see where you need to study.
- Solutions will be made available a little after the problems are released.

# Preparing for the Exam

- We've released a handout (Handout 16) containing advice about how to prepare for the exam, along with advice from previous CS103 students.
- Read over it… There's some good advice in there!

# Your Questions

“How can a non-CS major be competitive for the coterm in terms of classes, gpa, and recommendations?”

The CS coterm program is designed to be open to everyone and the admit rate is very, very high. You're not competing against anyone else. You just need to have a solid CS GPA (aim for 3.7 or higher if you can!) and have enough of a track record in CS classes that the admissions folks think that you'll do well in our later classes.

In terms of recs: don't feel afraid to ask for a DWIC (Did Well In Class) recommendation. We get requests like that all the time!

“I realized that I want to work as a software engineer after it was too late to change majors. How can I work in CS without a CS degree?”

The coterm is a pretty good option. ☺

Barring that, you'll want to find a structured framework for learning (a coding bootcamp, or freelance work, or a good mentor, etc.) so that you're in an environment where you're surrounded by folks you can learn from. Good life advice: you learn the most when you're surrounded by people who are way more accomplished than you!

Also, learn both computer science and software engineering. Try to learn how to program things like websites and apps and also how to think recursively, analyze algorithms, etc.

“Given the evidence from physicists and computer scientists, do you believe that we could be living in a simulation?”

I place the “simulation hypothesis” in as part of a long tradition of “using some new modality of inquiry to speculate about the nature of reality,” along the lines of Mu‘tazila theology (synthesizing Islam with the Greek rationalist tradition) or deism (synthesizing Christianity with Enlightenment thought).

The main arguments I’ve seen advanced for the simulation hypothesis are really, really shaky. I see many of the arguments – like wondering why information-theoretic concepts show up in physics – as great launching points for further study, but dispute the conclusions as well as the underlying line of reasoning. Just my two cents’. ☺

Back to CS103!

# Unequal Cardinalities

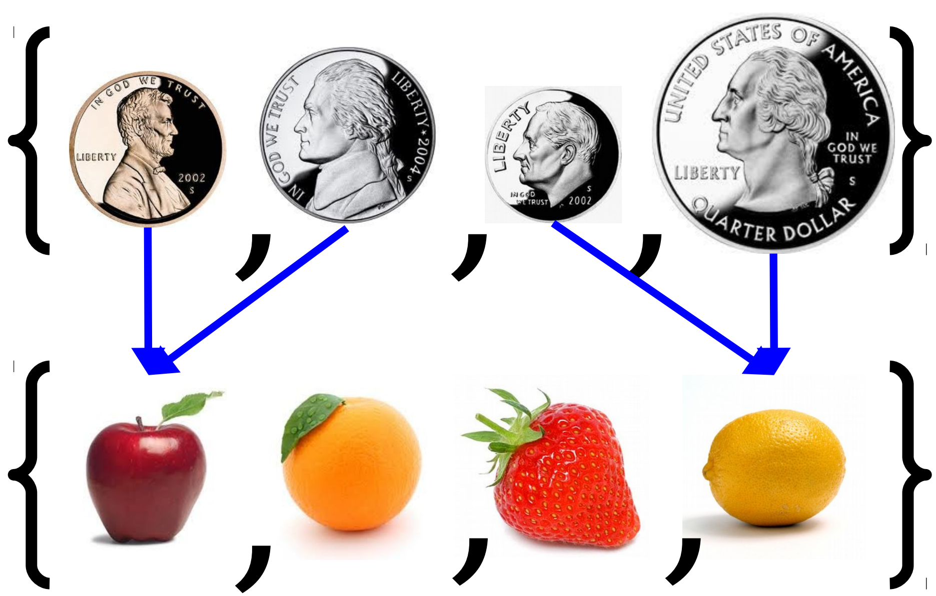
- Recall: $|A| = |B|$ if the following statement is true:

**There exists a bijection $f : A \to B$**

- What does it mean for $|A| \neq |B|$ to be true?

**Every function $f : A \to B$ is not a bijection.**

- This is a strong statement! To prove $|A| \neq |B|$, we need to show that *no possible function* from $A$ to $B$ can be injective and surjective.



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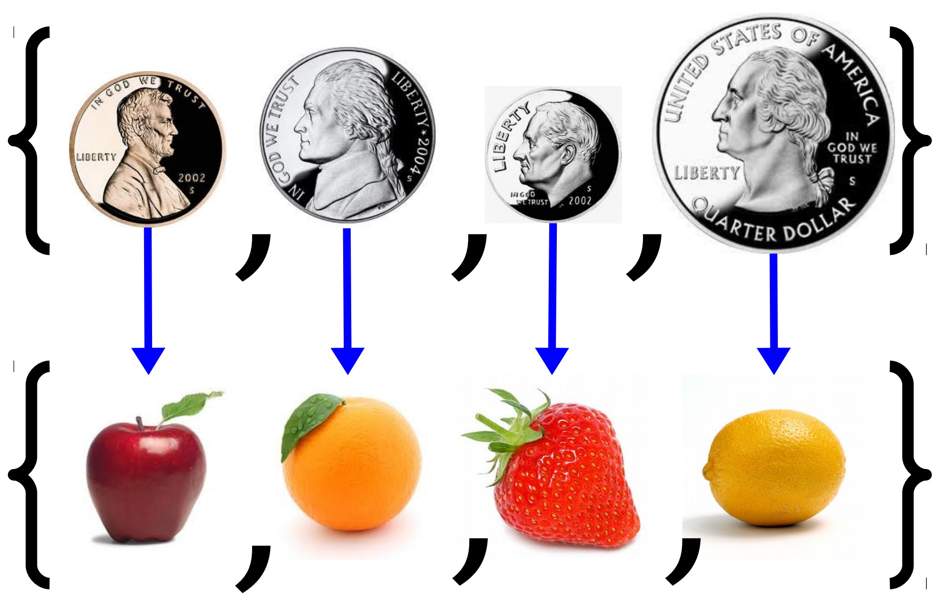

What is the relation between $|\mathbb{N}|$ and $|\mathbb{R}|$?

# Some Starting Assumptions

- Every real number can be written as an infinitely-long decimal number.
  - 0.5 = 0.500000000…
  - $^{\pi}/_{10}$ = 0.314159265…
- Two real numbers are different if their representations disagree at any point.
- We'll conveniently ignore issues of multiple representations of the same number (like 0.50000… = 0.499999…) for now.

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$

***Theorem:*** $|\mathbb{N}| \neq |[0, 1)|$

# Our Goal

- We need to show the following:

  **No function $f : \mathbb{N} \to [0, 1)$ is bijective**

- To prove it, we will do the following:
  - Choose an arbitrary function $f : \mathbb{N} \to [0, 1)$.
  - Show that $f$ cannot be a surjection by finding some $d \in [0, 1)$ that is not mapped to by $f$.
  - Conclude that this arbitrary function $f$ is not a bijection, so no bijections from $\mathbb{N}$ to $[0, 1)$ exist.

# The Intuition

- Suppose $f : \mathbb{N} \to [0, 1)$.
- We can then list off an infinite sequence of real numbers

$$f(0), f(1), f(2), f(3), \ldots .$$

We will show that we can always find a real number $d \in [0, 1)$ such that

**If $n \in \mathbb{N}$, then $f(n) \neq d$.**

# Rewriting Our Constraints

- Our goal is to find some $d \in [0, 1)$ such that

$$\textbf{If } \boldsymbol{n \in \mathbb{N}}\textbf{, then } \boldsymbol{f(n) \neq d}.$$

- In other words, we want to pick $d$ such that

$$\boldsymbol{f(0) \neq d}$$

$$\boldsymbol{f(1) \neq d}$$

$$\boldsymbol{f(2) \neq d}$$

$$\boldsymbol{f(3) \neq d}$$

$$\boldsymbol{\ldots}$$

# The Critical Insight

- **Key Idea:** Build the real number $d$ out of infinitely many "pieces," with one piece for each natural number.
  - Choose the $0^{\text{th}}$ piece such that $f(0) \neq d$.
  - Choose the $1^{\text{st}}$ piece such that $f(1) \neq d$.
  - Choose the $2^{\text{nd}}$ piece such that $f(2) \neq d$.
  - Choose the $3^{\text{rd}}$ piece such that $f(3) \neq d$.
  - …
- This "frankenreal" is specifically constructed so that $f(n) \neq d$ for all $n \in \mathbb{N}$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $\longleftrightarrow$ | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | $\longleftrightarrow$ | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | $\longleftrightarrow$ | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | $\longleftrightarrow$ | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | $\longleftrightarrow$ | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | $\longleftrightarrow$ | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | $\longleftrightarrow$ | ... | ... | ... | ... | ... | ... | ... | ... |

| | | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | … |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $\longleftrightarrow$ | 0. | 6 | 7 | 5 | 3 | 0 | 9 | … |
| 1 | $\longleftrightarrow$ | 0. | 1 | 4 | 1 | 5 | 9 | 2 | … |
| 2 | $\longleftrightarrow$ | 0. | 1 | 2 | 3 | 5 | 8 | 3 | … |
| 3 | $\longleftrightarrow$ | 0. | 0 | 0 | 0 | 0 | 0 | 0 | … |
| 4 | $\longleftrightarrow$ | 0. | 7 | 1 | 8 | 2 | 8 | 1 | … |
| 5 | $\longleftrightarrow$ | 0. | 6 | 1 | 8 | 0 | 3 | 4 | … |
| … | $\longleftrightarrow$ | … | … | … | … | … | … | … | … |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 6 | 4 | 3 | 0 | 8 | 4 | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

Set all nonzero values to 0 and all 0s to 1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | … |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | … |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | … |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | … |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | … |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | … |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | … |
| … | … | … | … | … | … | … | … | … |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | … |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

Which natural number is paired with this real number?

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

Which natural number is paired with this real number?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | … |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | … |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | … |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | … |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | … |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | … |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | … |
| … | … | … | … | … | … | … | … | … |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | … |

Which natural number is paired with this real number?

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

Which natural number is paired with this real number?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | … |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | … |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | … |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | … |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | … |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | … |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | … |
| … | … | … | … | … | … | … | … | … |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | … |

Which natural number is paired with this real number?

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

Which natural number is paired with this real number?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

Which natural number is paired with this real number?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 6 | 7 | 5 | 3 | 0 | 9 | ... |
| 1 | 0. | 1 | 4 | 1 | 5 | 9 | 2 | ... |
| 2 | 0. | 1 | 2 | 3 | 5 | 8 | 3 | ... |
| 3 | 0. | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0. | 7 | 1 | 8 | 2 | 8 | 1 | ... |
| 5 | 0. | 6 | 1 | 8 | 0 | 3 | 4 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

Which natural number is paired with this real number?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0 | 0 | 0 | 1 | 0 | 0 | ... |

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

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|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | |
|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | … |

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| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | | |
|---|---|---|---|---|---|---|
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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

***f*(0)**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | … |

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

***f*(0)[0]**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | … |

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

***f*(0)[0]**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | $\boxed{3}$ | 0 | 7 | 8 | 0 | … |

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

***f*(1)**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | $\boxed{0}$ | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

$\boldsymbol{f(1)[1]}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

**$f(1)[1]$**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | $\boxed{0}$ | 7 | 8 | 0 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | $\boxed{7}$ | 8 | 0 | … |

$f(2)[2]$

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

***f*****(3)[3]**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | $\boxed{8}$ | 0 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | |
|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | $\boxed{0}$ | … |

***f*(4)[4]**

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

***f*(*n*)[*n*]**

| | | | | | |
|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | $\boxed{…}$ |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 3 | 0 | 7 | 8 | 0 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 0 | 1 | 0 | 0 | 1 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | ... |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | ... |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | ... |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | ... |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | ... |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | ... |
| ... | 0. | ... | ... | ... | ... | ... | ... |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 0 | 1 | $\boxed{0}$ | 0 | 1 | ... |

**0 if $f(2)[2] \neq 0$**
**1 if $f(2)[2] = 0$**

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

**0 if $f(3)[3] \neq 0$**
**1 if $f(3)[3] = 0$**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 0 | 1 | 0 | $\boxed{0}$ | 1 | … |

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$.

| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 0 | 1 | 0 | 0 | $\boxed{1}$ | … |

**0 if $f(4)[4] \neq 0$**
**1 if $f(4)[4] = 0$**

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

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| | | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | … |
|---|---|---|---|---|---|---|---|
| 0 | 0. | 3 | 1 | 4 | 1 | 5 | … |
| 1 | 0. | 1 | 0 | 2 | 0 | 3 | … |
| 2 | 0. | 8 | 6 | 7 | 5 | 3 | … |
| 3 | 0. | 2 | 7 | 1 | 8 | 2 | … |
| 4 | 0. | 0 | 0 | 0 | 0 | 0 | … |
| … | 0. | … | … | … | … | … | … |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0. | 0 | 1 | 0 | 0 | 1 | … |

**0 if $f(n)[n] \neq 0$**
**1 if $f(n)[n] = 0$**

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$. Now, let $d$ be the real number whose integer part is zero and whose decimal digits obey the following rules:

$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

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Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

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Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$.

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

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Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$. To see why, consider any $n \in \mathbb{N}$ and look at $f(n)[n]$.

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$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$. To see why, consider any $n \in \mathbb{N}$ and look at $f(n)[n]$. There are two possibilities:

*Case 1:* $f(n)[n] = 0$.

*Case 2:* $f(n)[n] \neq 0$.

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$. Now, let $d$ be the real number whose integer part is zero and whose decimal digits obey the following rules:

$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$. To see why, consider any $n \in \mathbb{N}$ and look at $f(n)[n]$. There are two possibilities:

*Case 1:* $f(n)[n] = 0$. By construction $d[n] = 1$, so $f(n) \neq d$.

*Case 2:* $f(n)[n] \neq 0$.

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$. Now, let $d$ be the real number whose integer part is zero and whose decimal digits obey the following rules:

$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$. To see why, consider any $n \in \mathbb{N}$ and look at $f(n)[n]$. There are two possibilities:

*Case 1:* $f(n)[n] = 0$. By construction $d[n] = 1$, so $f(n) \neq d$.

*Case 2:* $f(n)[n] \neq 0$. By construction $d[n] = 0$, so $f(n) \neq d$.

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

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$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$. To see why, consider any $n \in \mathbb{N}$ and look at $f(n)[n]$. There are two possibilities:

*Case 1:* $f(n)[n] = 0$. By construction $d[n] = 1$, so $f(n) \neq d$.

*Case 2:* $f(n)[n] \neq 0$. By construction $d[n] = 0$, so $f(n) \neq d$.

Because $f(n) \neq d$ for any $n \in \mathbb{N}$, we see $f$ is not surjective, so $f$ is not a bijection.

***Theorem:*** $|\mathbb{N}| \neq |\mathbb{R}|$.

***Proof:*** We will prove that $|\mathbb{N}| \neq |[0, 1)|$. Since $|[0, 1)| = |\mathbb{R}|$, this shows that $|\mathbb{N}| \neq |\mathbb{R}|$. To show that $|\mathbb{N}| \neq |[0, 1)|$, we will prove that there is no bijective function $f : \mathbb{N} \to [0, 1)$. To do so, let $f : \mathbb{N} \to [0, 1)$ be an arbitrary function. We will show that it is not surjective.

Let's introduce some new notation. For any $r \in [0, 1)$, let $r[n]$ be the $n$th decimal digit in the decimal representation of $r$. Now, let $d$ be the real number whose integer part is zero and whose decimal digits obey the following rules:

$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $d$'s integer part is zero, we know that $d \in [0, 1)$.

We claim that $f(n) \neq d$ for any $n \in \mathbb{N}$. To see why, consider any $n \in \mathbb{N}$ and look at $f(n)[n]$. There are two possibilities:

*Case 1:* $f(n)[n] = 0$. By construction $d[n] = 1$, so $f(n) \neq d$.

*Case 2:* $f(n)[n] \neq 0$. By construction $d[n] = 0$, so $f(n) \neq d$.

Because $f(n) \neq d$ for any $n \in \mathbb{N}$, we see $f$ is not surjective, so $f$ is not a bijection. Since $f$ was chosen arbitrarily, this means that there is no bijection $f : \mathbb{N} \to [0, 1)$, so $|\mathbb{N}| \neq |[0, 1)|$.

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# An Interesting Historical Aside

- The proof we just covered was first discovered by Georg Cantor.
- Interestingly, this was *not* Cantor's first proof that $|\mathbb{N}| \neq |\mathbb{R}|$. He first developed a different proof based on converging sequences of real numbers.
- Curious? Come talk to us after class!

# The Guide to Cantor's Theorem

- We've put together a Guide to Cantor's Theorem up on the course website that contains a complete and rigorous proof of Cantor's theorem, along with an explanation of the key steps.
- You'll need to read this before doing PS4.
- Feel free to get a jump on it now, or perendinate until later. ☺

# Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Connectivity***
  - Finding clusters in graphs.
- ***Planar Graphs***
  - Definitions from pictures!
- ***Graph Coloring***
  - A famous problem in graph theory.