

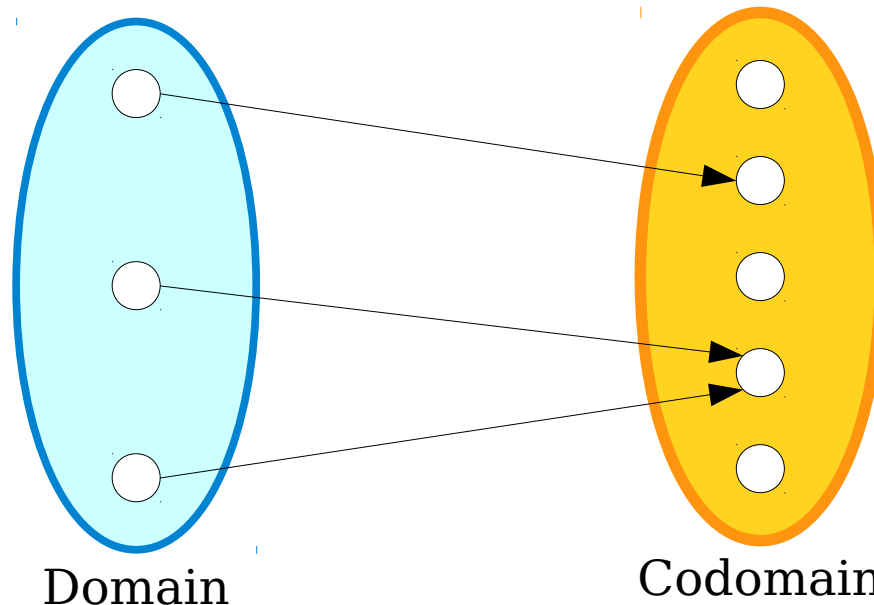
# Cardinality

Recap from Last Time

# Domains and Codomains

- Every function  $f$  has two sets associated with it: its **domain** and its **codomain**.
- A function  $f$  can only be applied to elements of its domain. For any  $x$  in the domain,  $f(x)$  belongs to the codomain.
- We write  $f : A \rightarrow B$  to indicate that  $f$  is a function whose domain is  $A$  and whose codomain is  $B$ .

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be producible.

# Function Composition

- If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are functions, the **composition of  $f$  and  $g$** , denoted  $g \circ f$ , is a function
  - whose domain is  $A$ ,
  - whose codomain is  $C$ , and
  - which is evaluated as  $(g \circ f)(x) = g(f(x))$ .

# Injective Functions

- A function  $f : A \rightarrow B$  is called **injective** (or **one-to-one**) if each element of the codomain has at most one element of the domain that maps to it.
  - A function with this property is called an **injection**.
- Formally,  $f : A \rightarrow B$  is an injection if this FOL statement is true:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

*(“If the inputs are different, the outputs are different”)*

- Equivalently:

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

*(“If the outputs are the same, the inputs are the same”)*

- **Theorem:** The composition of two injections is an injection.

# Surjective Functions

- A function  $f : A \rightarrow B$  is called **surjective** (or **onto**) if each element of the codomain is “covered” by at least one element of the domain.
  - A function with this property is called a **surjection**.
- Formally,  $f : A \rightarrow B$  is a surjection if this FOL statement is true:

$$\forall b \in B. \exists a \in A. f(a) = b$$

*(“For every possible output, there's at least one possible input that produces it”)*

- **Theorem:** The composition of two surjections is a surjection.

# Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijjective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- ***Theorem***: The composition of two bijections is a bijection.
- ***Theorem***: For any bijection  $f : A \rightarrow B$ , there is a function  $f^{-1} : B \rightarrow A$  that is the ***inverse*** of  $f$ , and  $f^{-1}$  is also a bijection.

# Where We Are

- We now know
  - what an injection, surjection, and bijection are;
  - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
  - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason...



New Stuff!

# Cardinality Revisited

# Cardinality

- Recall (*from our first lecture!*) that the **cardinality** of a set is the number of elements it contains.
- If  $S$  is a set, we denote its cardinality by  $|S|$ .
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced **infinite cardinals** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

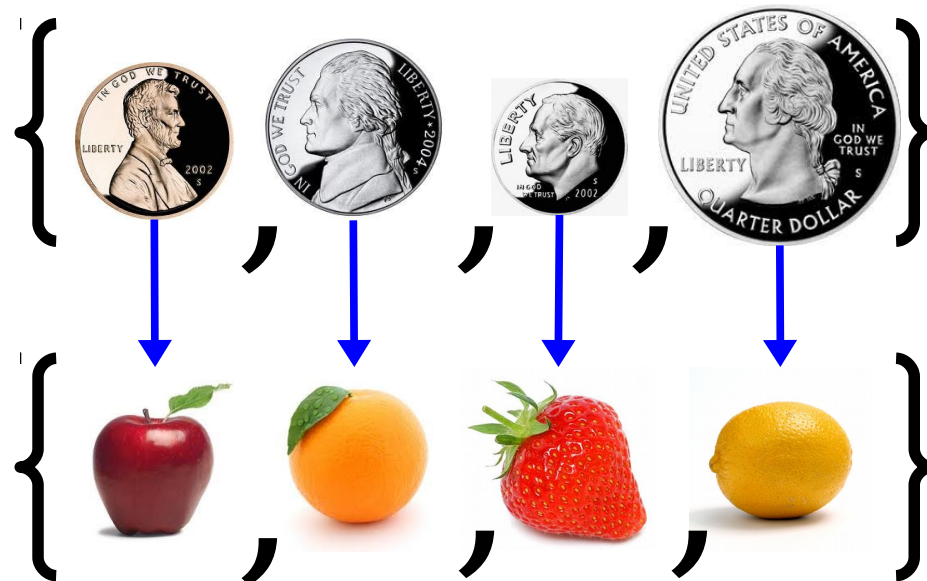
# Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is  $\aleph_0$ ?
  - (Take Math 161 for an answer!)
- **Idea:** Define cardinality as a *relation* between two sets rather than an absolute quantity.

# Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

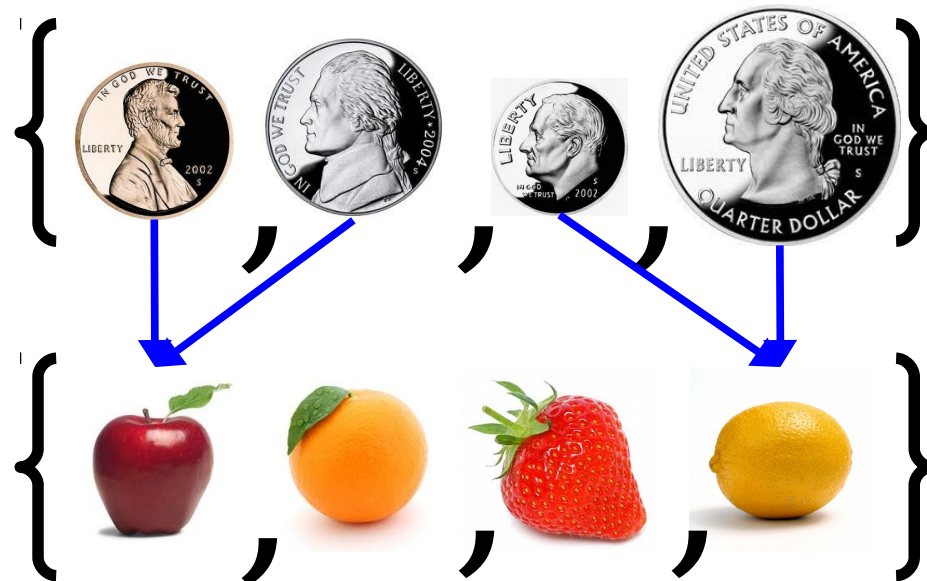
**$|S| = |T|$  if there exists a *bijection*  $f : S \rightarrow T$**



# Comparing Cardinalities

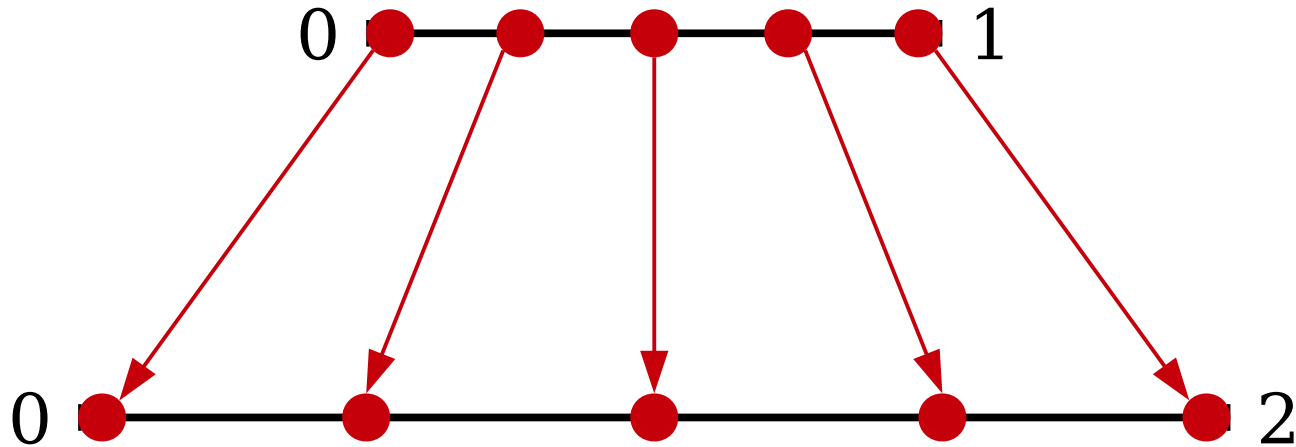
- Here is the formal definition of what it means for two sets to have the same cardinality:

**$|S| = |T|$  if there exists a *bijection*  $f : S \rightarrow T$**



# Fun with Cardinality

# Home on the Range



$$f : [0, 1] \rightarrow [0, 2]$$
$$f(x) = 2x$$



**Theorem:**  $|[0, 1]| = |[0, 2]|$

**Proof:** Consider the function  $f : [0, 1] \rightarrow [0, 2]$  defined as  $f(x) = 2x$ . We will prove that  $f$  is a bijection.

First, we will show that  $f$  is a well-defined function. Choose any  $x \in [0, 1]$ . This means that  $0 \leq x \leq 1$ , so we know that  $0 \leq 2x \leq 2$ . Consequently, we see that  $0 \leq f(x) \leq 2$ , so  $f(x) \in [0, 2]$ .

Next, we'll show that  $f$  is injective. Pick any  $x_1, x_2 \in [0, 1]$  where  $f(x_1) = f(x_2)$ . We will show that  $x_1 = x_2$ . To see this, notice that since  $f(x_1) = f(x_2)$ , we see that  $2x_1 = 2x_2$ , which in turn tells us that  $x_1 = x_2$ , as required.

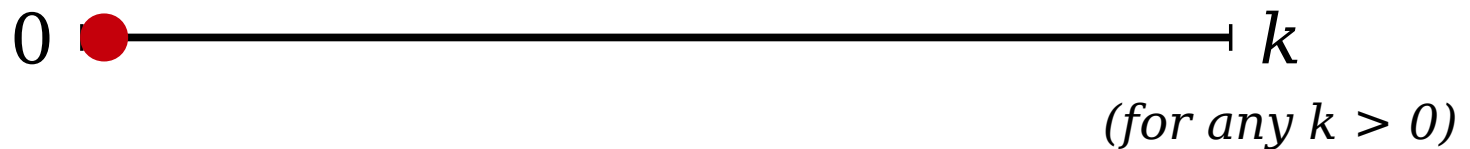
Finally, we will show that  $f$  is surjective. To do so, consider any  $y \in [0, 2]$ . We'll show that there is some  $x \in [0, 1]$  where  $f(x) = y$ .

Let  $x = y/2$ . Since  $y \in [0, 2]$ , we know  $0 \leq y \leq 2$ , and therefore that  $0 \leq y/2 \leq 1$ . We picked  $x = y/2$ , so we know that  $0 \leq x \leq 1$ , which in turn means  $x \in [0, 1]$ . Moreover, notice that

$$f(x) = 2x = 2(y/2) = y,$$

so  $f(x) = y$ , as required. ■

# Home on the Range

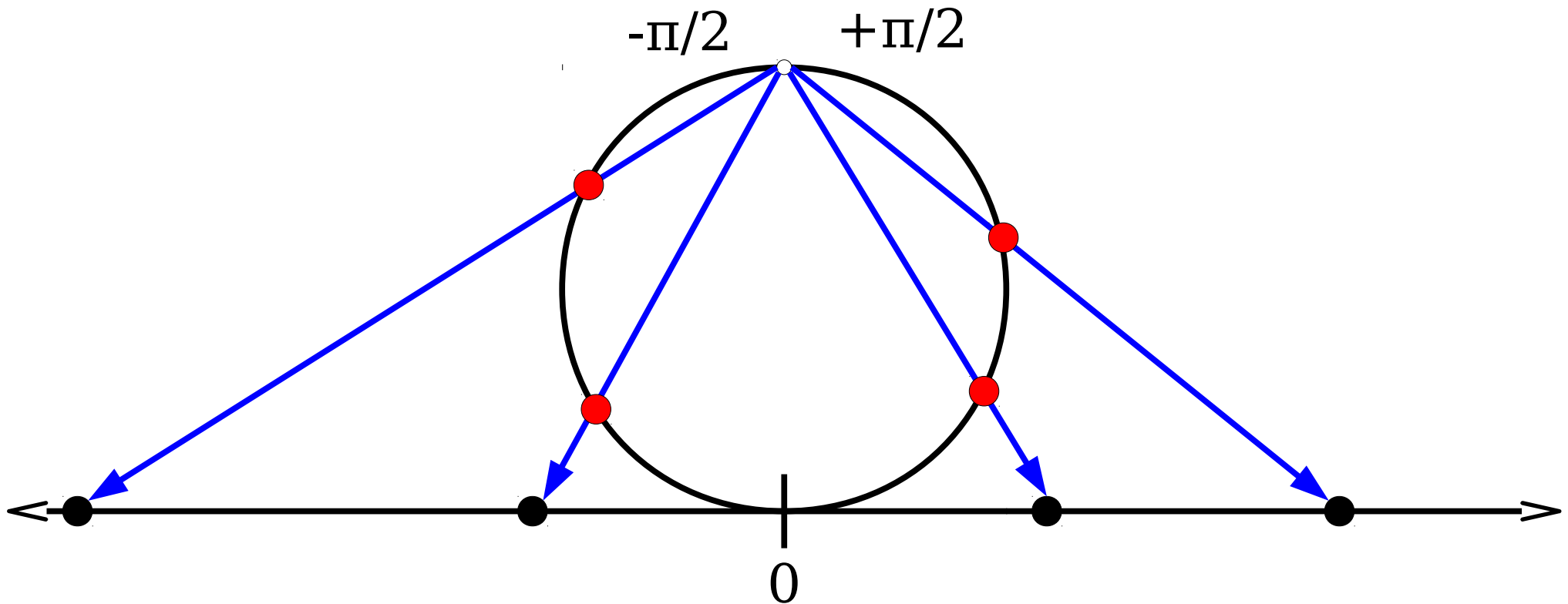


$$f : [0, 1] \rightarrow [0, k]$$
$$f(x) = kx$$

This means that *cardinality* (how many points there are) is a different idea than *mass* (how much those points weight). Look into *measure theory* if you're curious to learn more!

And one more example, just for funzies.

# Put a Ring On It



$$f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$$
$$f(x) = \tan x$$

$$|(-\pi/2, \pi/2)| = |\mathbb{R}|$$

# Some Properties of Cardinality

**Theorem:** For any set  $A$ , we have  $|A| = |A|$ .

**Proof:** Consider any set  $A$ , and let  $f : A \rightarrow A$  be the function defined as  $f(x) = x$ . We will prove that  $f$  is a bijection.

First, we'll show that  $f$  is injective. Pick any  $x_1, x_2 \in A$  where  $f(x_1) = f(x_2)$ . We need to show that  $x_1 = x_2$ . Since  $f(x_1) = f(x_2)$ , we see by definition of  $f$  that  $x_1 = x_2$ , as required.

Next, we'll show that  $f$  is surjective. Consider any  $y \in A$ . We will prove that there is some  $x \in A$  where  $f(x) = y$ . Pick  $x = y$ . Then  $f(x) = x = y$ , as required. ■

**Theorem:** If  $A$ ,  $B$ , and  $C$  are sets where  $|A| = |B|$  and  $|B| = |C|$ , then  $|A| = |C|$ .

**Proof:** Consider any sets  $A$ ,  $B$ , and  $C$  where  $|A| = |B|$  and  $|B| = |C|$ . We need to prove that  $|A| = |C|$ . To do so, we need to show that there is a bijection from  $A$  to  $C$ .

Since  $|A| = |B|$ , we know that there is a some bijection  $f : A \rightarrow B$ . Similarly, since  $|B| = |C|$  we know that there is at least one bijection  $g : B \rightarrow C$ .

Consider the function  $g \circ f : A \rightarrow C$ . Since  $g$  and  $f$  are bijections and the composition of two bijections is a bijection, we see that  $g \circ f$  is a bijection from  $A$  to  $C$ . Thus  $|A| = |C|$ , as required. ■

***Great exercise:*** Prove that if  $A$  and  $B$  are sets where  $|A| = |B|$ , then  $|B| = |A|$ .



**Time-Out for Announcements!**

# Apply to Section Lead!

- Applications are currently open for section leader positions in Winter and Spring.
  - Already did CS106B/X? Deadline is this Thursday at 11:59PM.
  - Currently in CS106B/X? Deadline is November 2<sup>nd</sup> at 11:59PM.
- This is an amazing opportunity. You will meet all sorts of cool people, get better at public speaking, and get to share the excitement of computer science.
- Apply online at

**<https://cs198.stanford.edu/cs198/Apply.aspx>**

# Problem Sets

- The PS3 checkpoint was due at 2:30PM. We'll get it graded and returned to you by Wednesday.
- PS3 is due this Friday. You *can* use late days on it, but it might not be a good idea to do that given that the midterm is next Monday.
- PS2 solutions are now available! You should definitely read over them and make sure you understand the answers. We'll get graded PS2's back to you by Wednesday as well.

# Midterm Exam Logistics

- The first midterm exam is next **Monday, October 23<sup>rd</sup>**, from **7:00PM - 10:00PM**. Locations are divvied up by last (family) name:
  - Abb - Lop: Go to **Cubberly Auditorium**.
  - Mac - Zwa: Go to **Hewlett 200**.
- You're responsible for Lectures 00 - 05 and topics covered in PS1 - PS2. Later lectures (relations forward) and problem sets (PS3 onward) won't be tested here.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.
- Students with OAE accommodations: please contact us **immediately** if you haven't yet done so. We'll ping you about setting up alternate exams.

# Midterm Exam

- ***We want you to do well on this exam.*** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your “mathematical potential” or “innate mathematical ability.”

# Practice Midterm Exam

- To help you prepare for the midterm, we'll be holding a practice midterm exam on **Wednesday, October 18** from **7PM - 10PM** in **Hewlett 200**.
- The practice midterm exam is an actual midterm we gave out in a previous quarter. It's probably the best indicator of what you should expect to see.
- Course staff will be on hand to answer your questions.
- Can't make it? We'll release the practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!

# Extra Practice Problems

- We released a set of extra practice problems on Friday of last week. Solutions are now available.
- We strongly recommend working through these practice problems. They're a great way to get additional practice with the material and to see where you need to study.
- We've released two practice midterms today. Solutions will go out on Wednesday. Use these resources strategically!

# Preparing for the Exam

- We've released a handout (Handout 20) containing advice about how to prepare for the exam, along with advice from previous CS103 students.
- Read over it... There's some good advice in there!



Your Questions

“Did you take CS 103 yourself as a student? If so, who taught the course then and what was your experience like?”

Yep! In fact, it's the course where I first learned discrete math! It was a ton of work – I remember putting in many, many hours working through problems, crossing things off, writing and rewriting things, etc. I learned a ton from the course!

I also remember getting the single most brutal midterm question I've ever had on the CS103 midterm, and so I've made a concerted effort not to write questions like that. 😊

“What is one non-CS class you recommend for everyone?”

If I had to pick just one, I'd recommend a creative writing class (English 90, English 91). It's amazing how valuable a skill this is. Putting things in writing is a great way to determine your limits of understanding, argumentation, and general intellectual clarity.

Alternatively, take whatever class you can find that's taught by someone who's about to retire. Those classes tend to be pretty special!

Back to CS103!

# Unequal Cardinalities

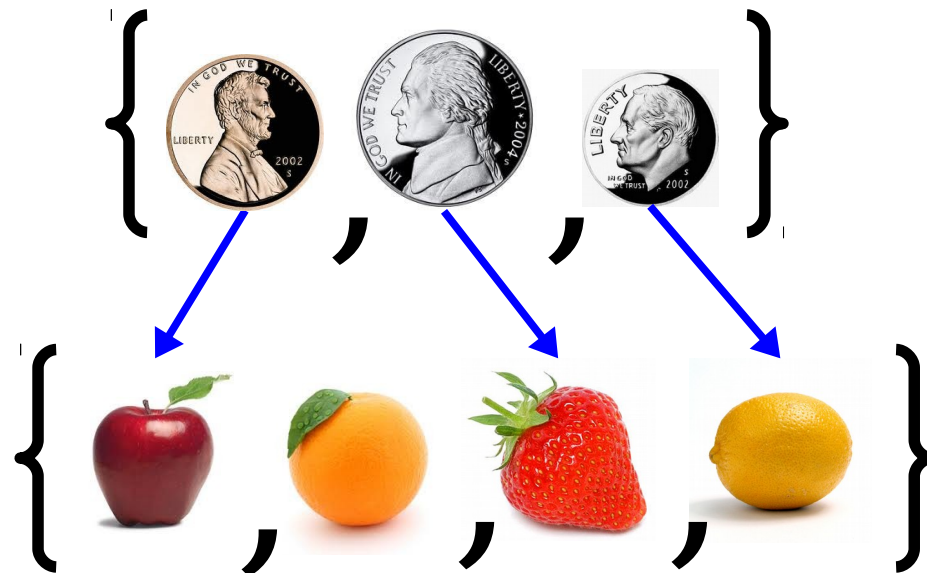
- Recall:  $|A| = |B|$  if the following statement is true:

**There exists a bijection  $f : A \rightarrow B$**

- What does it mean for  $|A| \neq |B|$  to be true?

**Every function  $f : A \rightarrow B$  is not a bijection.**

- This is a strong statement! To prove  $|A| \neq |B|$ , we need to show that *no possible function* from  $A$  to  $B$  can be injective and surjective.



# Cantor's Theorem Revisited

# Cantor's Theorem

- In our very first lecture, we sketched out a proof of *Cantor's theorem*, which says that

**If  $S$  is a set, then  $|S| < |\wp(S)|$ .**

- That proof was visual and pretty hand-wavy. Let's see if we can go back and formalize it!

# Where We're Going

- Today, we're going to formally prove the following result:

**If  $S$  is a set, then  $|S| \neq |\wp(S)|$ .**

- We've released an online Guide to Cantor's Theorem, which will go into *way* more depth than what we're going to see here.
- The goal for today will be to see how to start with our picture and turn it into something rigorous.
- On the next problem set, you'll explore the proof in more depth and see some other applications.



# The Roadmap

- We're going to prove this statement:

If  $S$  is a set, then  $|S| \neq |\wp(S)|$ .

- Here's how this will work:
  - Pick an arbitrary set  $S$ .
  - Pick an arbitrary function  $f : S \rightarrow \wp(S)$ .
  - Show that  $f$  is not surjective using a diagonal argument.
  - Conclude that there are no bijections from  $S$  to  $\wp(S)$ .
  - Conclude that  $|S| \neq |\wp(S)|$ .

*This is a drawing  
of our function  
 $f : S \rightarrow \wp(S)$ .*

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

*This is a drawing  
of our function  
 $f : S \rightarrow \wp(S)$ .*

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

*This is a drawing of our function  $f: S \rightarrow \wp(S)$ .*

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

*Flip all Y's to N's and vice-versa to get a new set*

N	Y	Y	Y	Y	N	...
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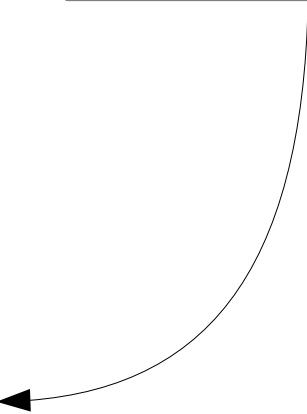
*This is a drawing of our function  $f : S \rightarrow \wp(S)$ .*

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...

<b>N</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
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*Which row in the table is paired with this set?*



*This is a drawing of our function  $f : S \rightarrow \wp(S)$ .*

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

*What set is this?*

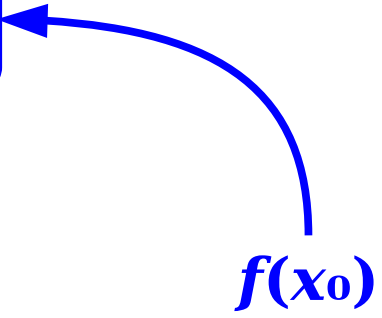
N	Y	Y	Y	Y	N	...
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*This is a drawing of our function  $f : S \rightarrow \wp(S)$ .*

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
---	---	---	---	---	---	-----



*This is a drawing of our function  $f : S \rightarrow \wp(S)$ .*

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_1$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
$x_2$	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_3$	<b>N</b>	<b>Y</b>	<b>N</b>	<b>N</b>	<b>Y</b>	<b>N</b>	...
$x_4$	<b>Y</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>N</b>	<b>Y</b>	...
$x_5$	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	...
...	...	...	...	...	...	...	...



$x_0 \in f(x_0)?$

<b>N</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>Y</b>	<b>N</b>	...
----------	----------	----------	----------	----------	----------	-----



This is a drawing of our function  $f : S \rightarrow \wp(S)$ .

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
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$x_0 \notin f(x_0)$ ?



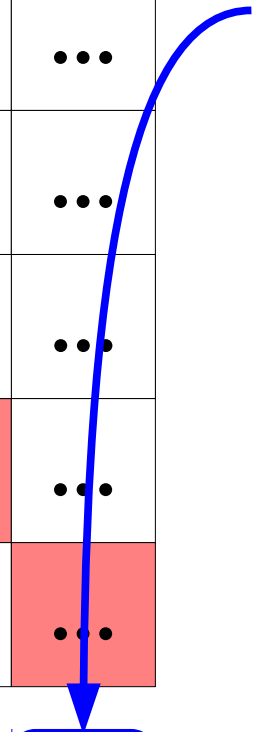
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	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

N	Y	Y	Y	Y	N	...
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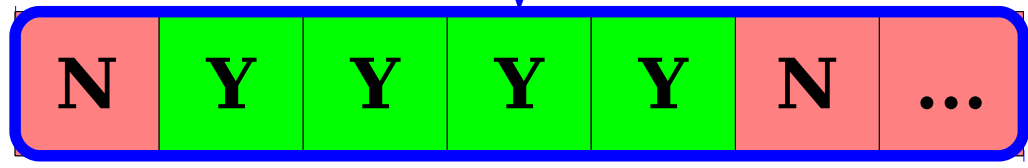
$x \notin f(x)?$



This is a drawing of our function  $f : S \rightarrow \wp(S)$ .

	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	...
$x_0$	Y	N	Y	N	Y	N	...
$x_1$	Y	N	N	Y	Y	N	...
$x_2$	N	N	N	N	Y	N	...
$x_3$	N	Y	N	N	Y	N	...
$x_4$	Y	N	N	N	N	Y	...
$x_5$	Y	Y	Y	Y	Y	Y	...
...	...	...	...	...	...	...	...

$\{ x \in S \mid x \notin f(x) \}$



# The Diagonal Set

- For any set  $S$  and function  $f : S \rightarrow \wp(S)$ , we can define a set  $D$  as follows:

$$D = \{ x \in S \mid x \notin f(x) \}$$

*(“The set of all elements  $x$  where  $x$  is not an element of the set  $f(x)$ .”)*

- This is a formalization of the set we found in the previous picture.
- Using this choice of  $D$ , we can formally prove that no function  $f : S \rightarrow \wp(S)$  is a bijection.

**Theorem:** If  $S$  is a set, then  $|S| \neq |\wp(S)|$ .

**Proof:** Let  $S$  be an arbitrary set. We will prove that  $|S| \neq |\wp(S)|$  by showing that there are no bijections from  $S$  to  $\wp(S)$ . To do so, choose an arbitrary function  $f : S \rightarrow \wp(S)$ . We will prove that  $f$  is not surjective.

Starting with  $f$ , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no  $y \in S$  such that  $f(y) = D$ . To do so, we proceed by contradiction. Suppose that there is some  $y \in S$  such that  $f(y) = D$ . By the definition of  $D$ , we know that

$$y \in D \text{ iff } y \notin f(y). \quad (2)$$

By assumption,  $f(y) = D$ . Combined with (2), this tells us

$$y \in D \text{ iff } y \notin D. \quad (3)$$

This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no  $y \in S$  such that  $f(y) = D$ , so  $f$  is not surjective. This means that  $f$  is not a bijection, and since our choice of  $f$  was arbitrary, we conclude that there are no bijections between  $S$  and  $\wp(S)$ . Thus  $|S| \neq |\wp(S)|$ , as required. ■

# The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor's theorem can be formalized in terms of surjectivity.

# Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!