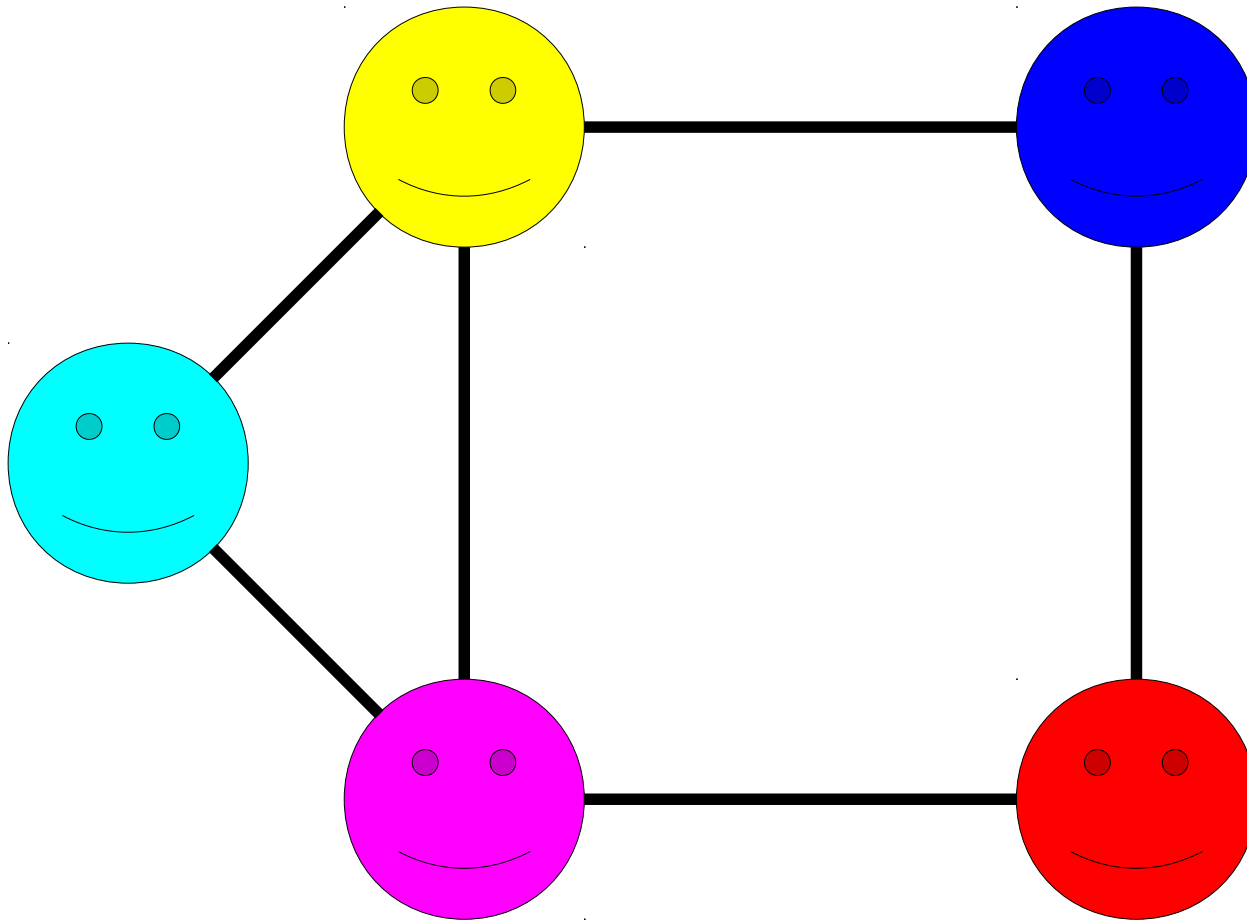


Graph Theory

Part Two

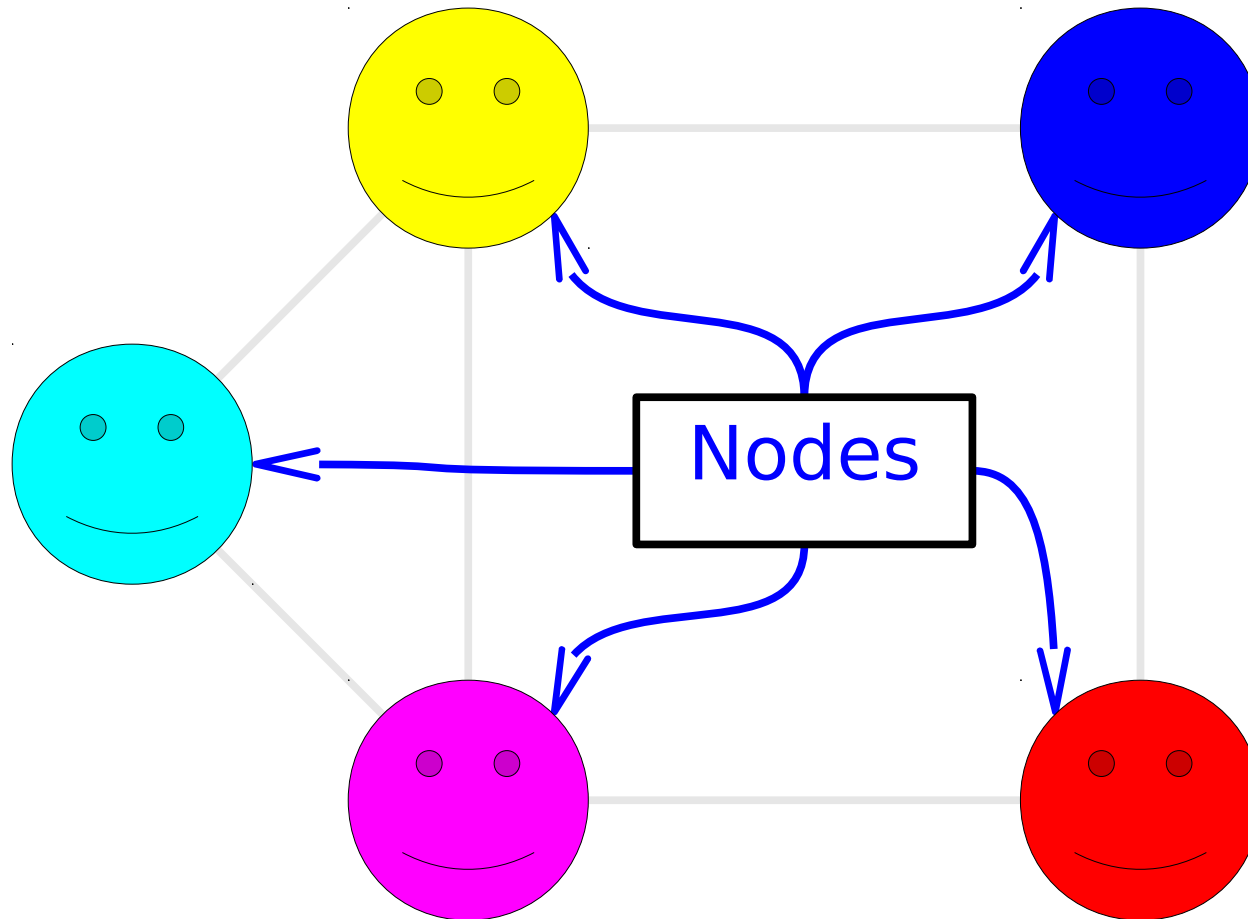
Recap from Last Time

A **graph** is a mathematical structure for representing relationships.



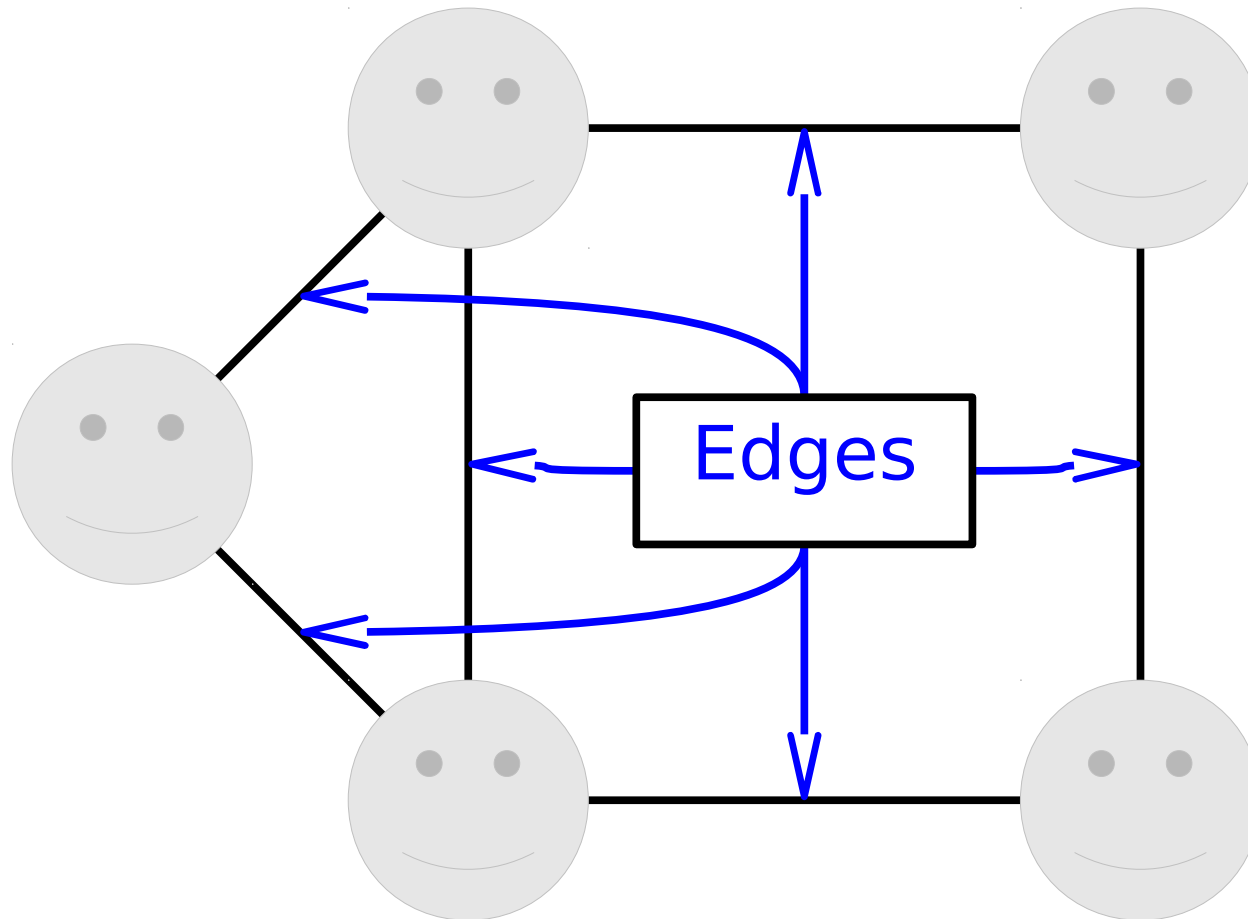
A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

A **graph** is a mathematical structure for representing relationships.



A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

A **graph** is a mathematical structure for representing relationships.



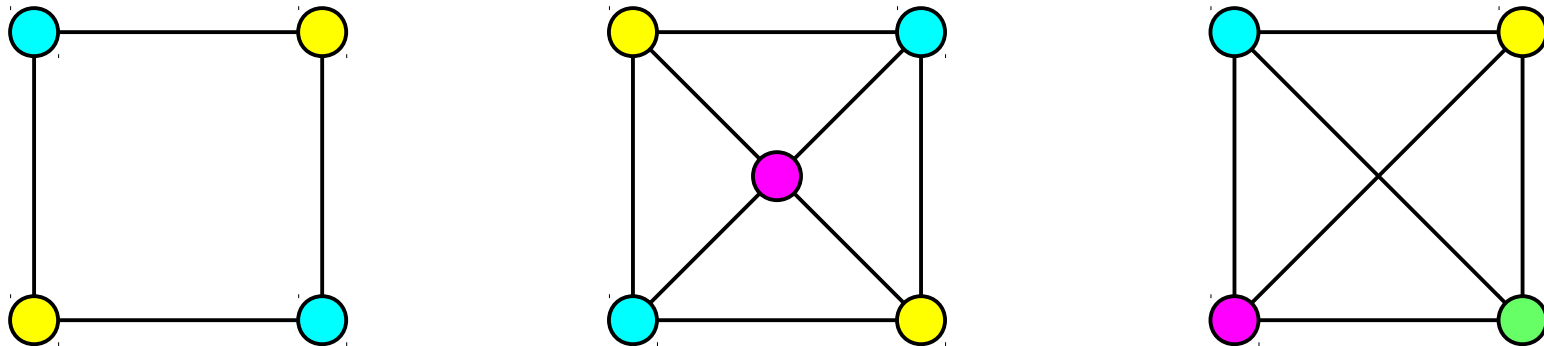
A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

Adjacency and Connectivity

- Two nodes in a graph are called ***adjacent*** if there's an edge between them.
- Two nodes in a graph are called ***connected*** if there's a path between them.
 - A path is a series of one or more nodes where consecutive nodes are adjacent.

k -Vertex-Colorings

- If $G = (V, E)$ is a graph, a **k -vertex-coloring** of G is a way of assigning colors to the nodes of G , using at most k colors, so that no two nodes of the same color are adjacent.



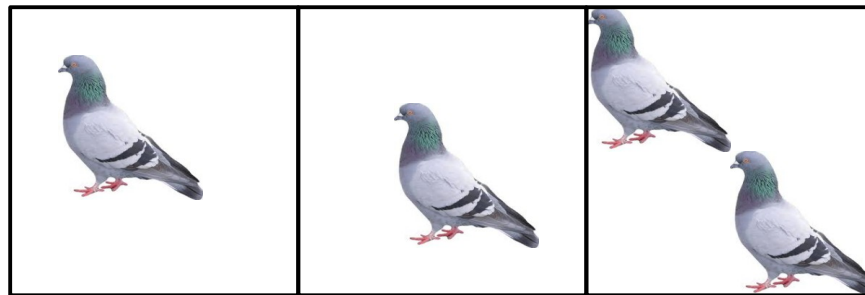
- The **chromatic number** of G , denoted $\chi(G)$, is the minimum number of colors needed in any k -coloring of G .
- Today, we're going to see several results involving coloring parts of graphs. They don't necessarily involve k -vertex-colorings of graphs, so feel free to ask for clarifications if you need them!

New Stuff!

The Pigeonhole Principle

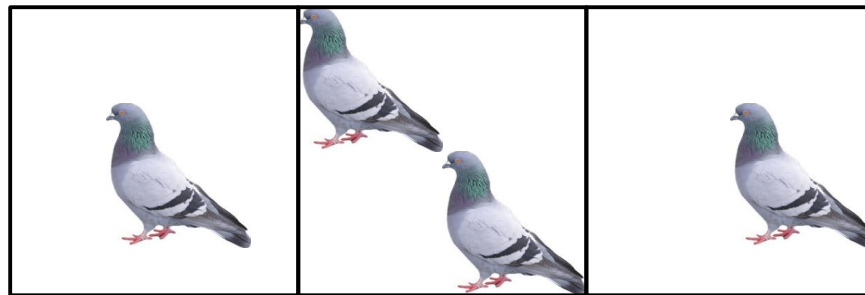
The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):***
If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



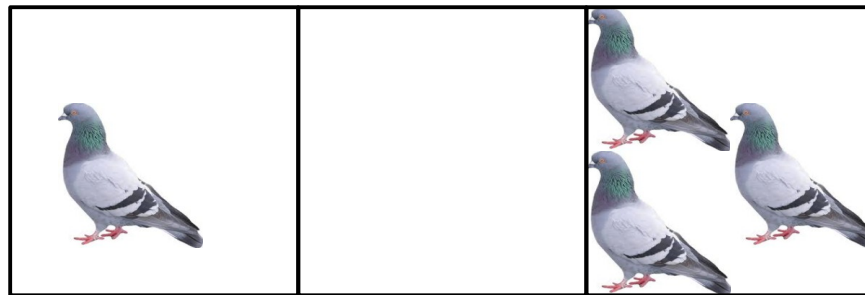
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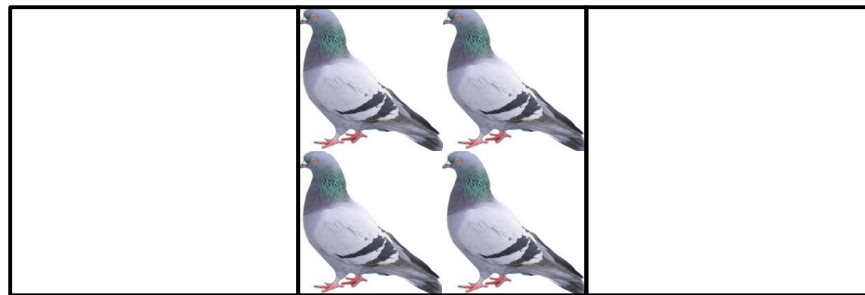
The Pigeonhole Principle

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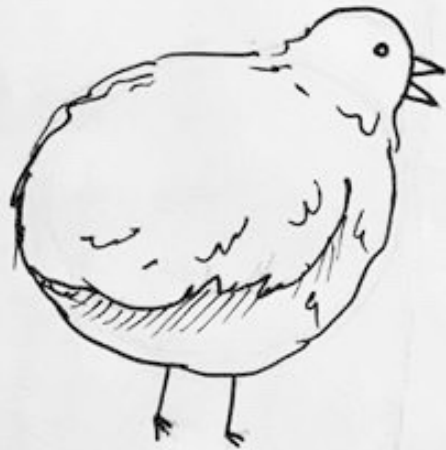


The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):***
If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



NO MORE
- PIGEON HOLES?!



$$m = 4, n = 3$$

Thanks to Amy Liu for this awesome drawing!

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
 - 366 possible birthdays (pigeonholes)
 - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
 - Maximum number of hairs ever found on a human head is no greater than 500,000.
 - There are over 800,000 people in San Francisco.

Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

Let A and B be finite sets (sets whose cardinalities are natural numbers) and assume $|A| > |B|$. How many of the following statements are true?

- (W) If $f : A \rightarrow B$, then f is injective.
- (X) If $f : A \rightarrow B$, then f is not injective.
- (Y) If $f : A \rightarrow B$, then f is surjective.
- (Z) If $f : A \rightarrow B$, then f is not surjective.

Answer at **[PollEv.com/cs103](https://www.pollEv.com/cs103)** or
text **CS103** to **22333** once to join, then a number.

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . There are m objects in total, so we know that

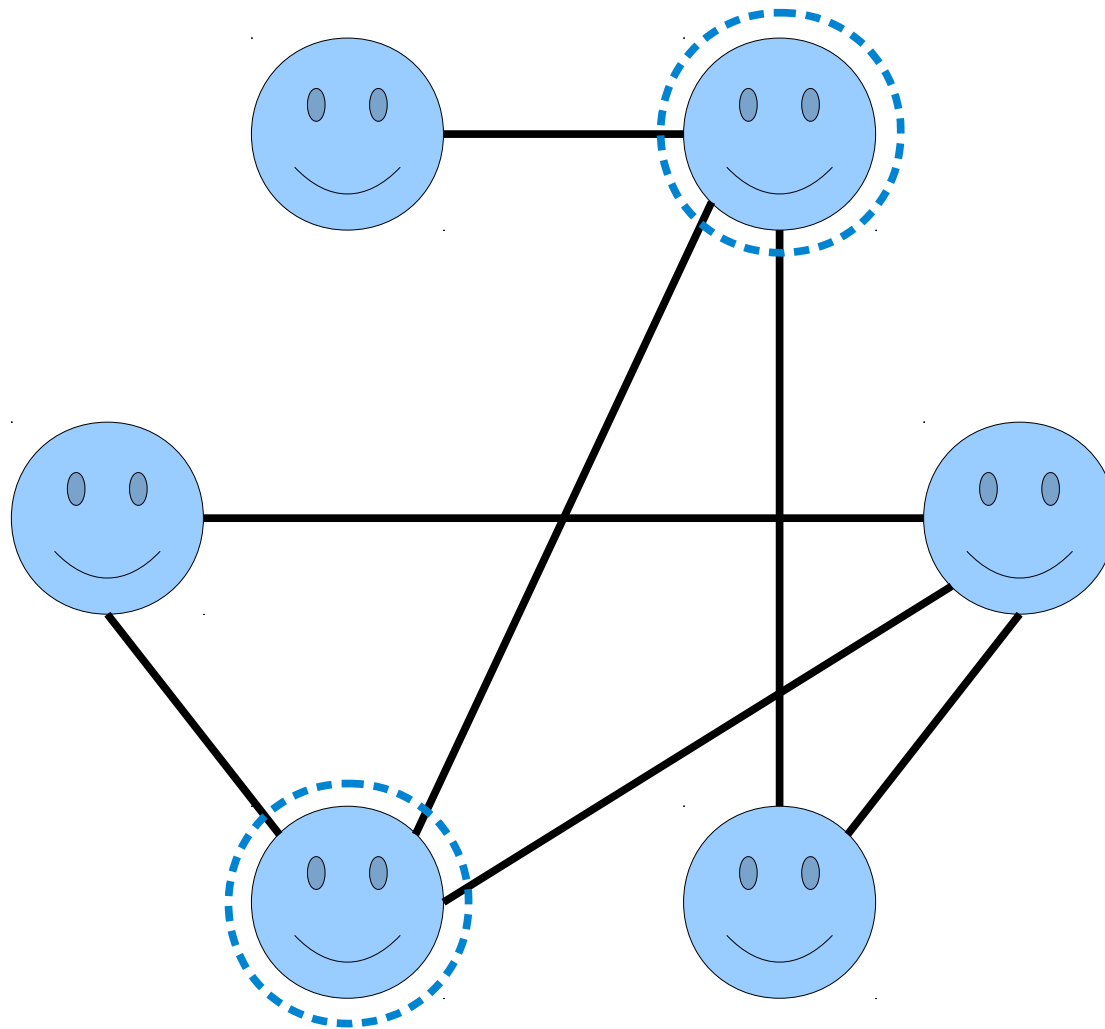
$$m = x_1 + x_2 + \dots + x_n.$$

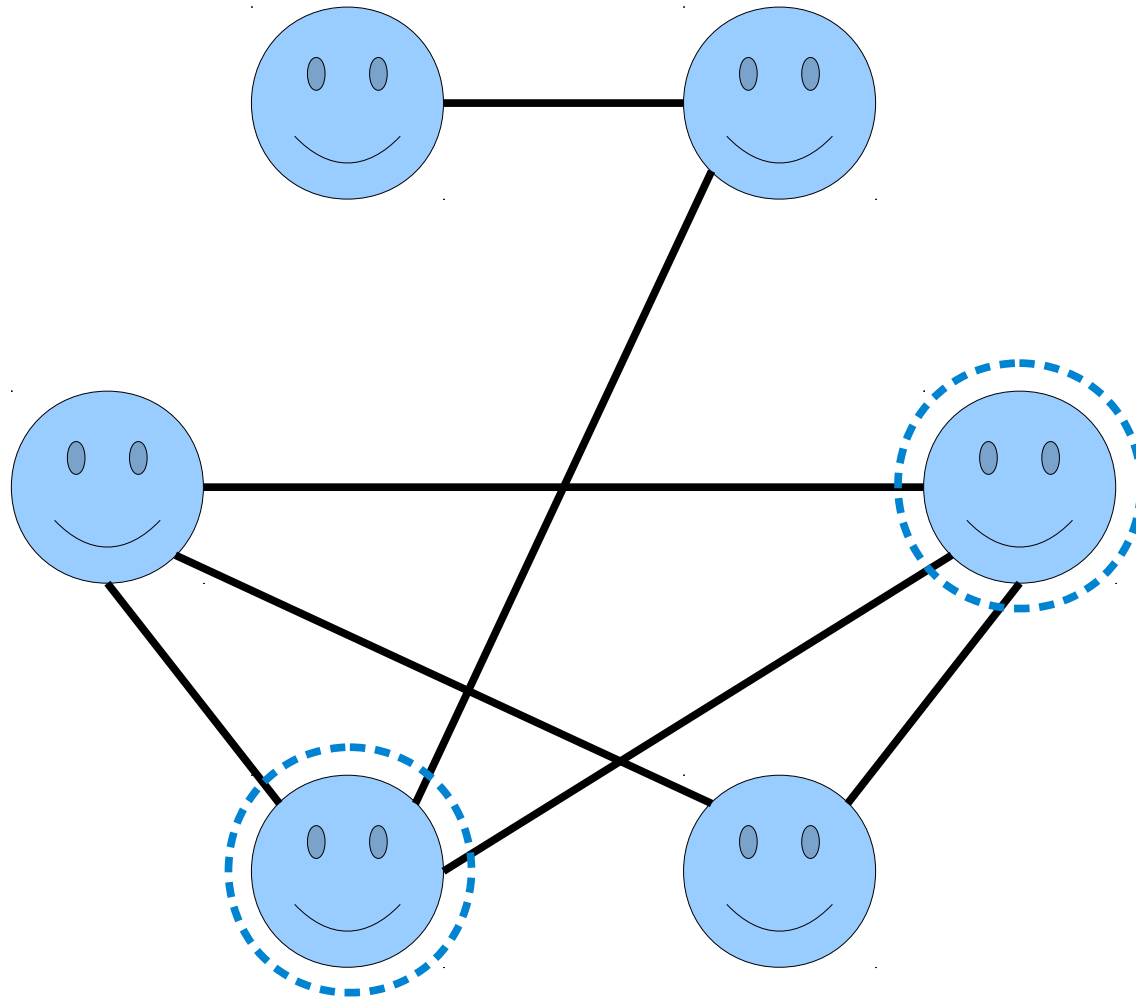
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i . This means that

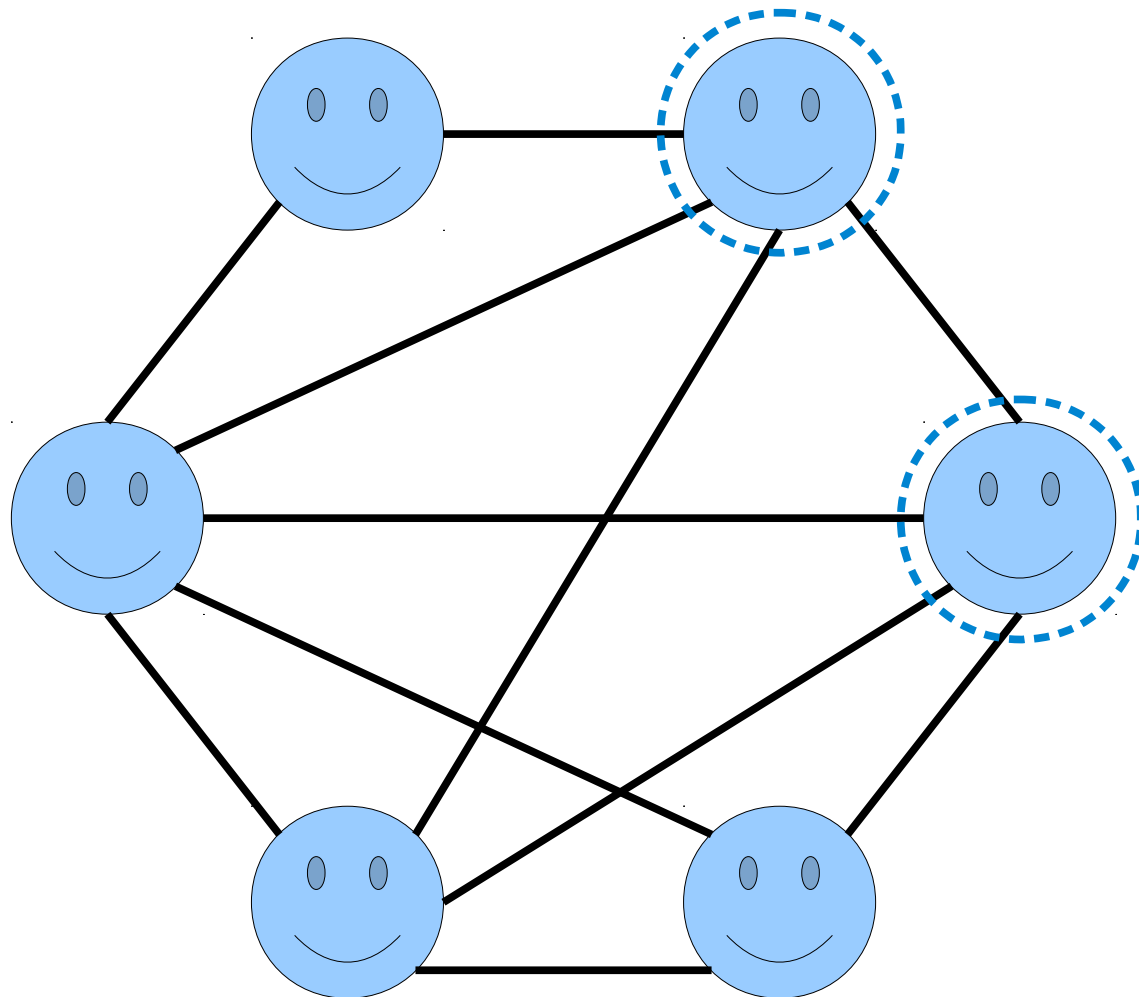
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &\leq 1 + 1 + \dots + 1 \quad (n \text{ times}) \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks

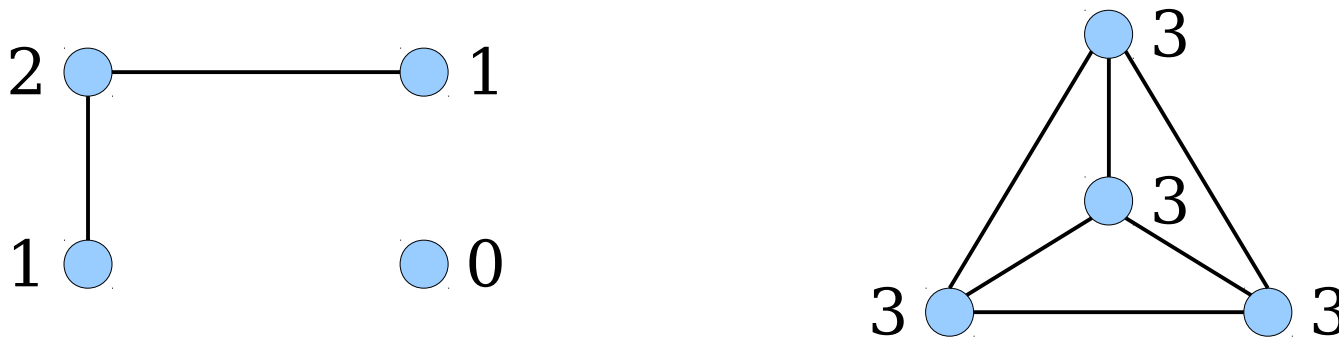




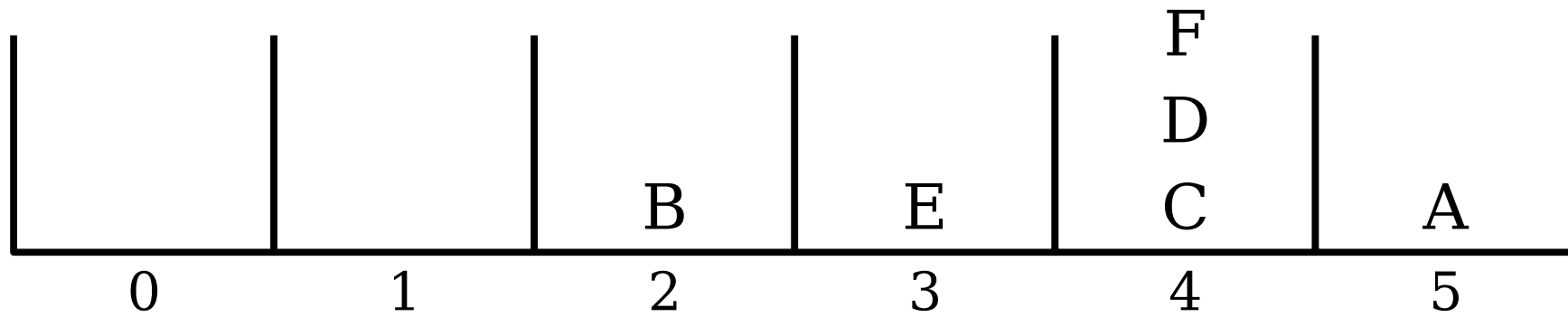
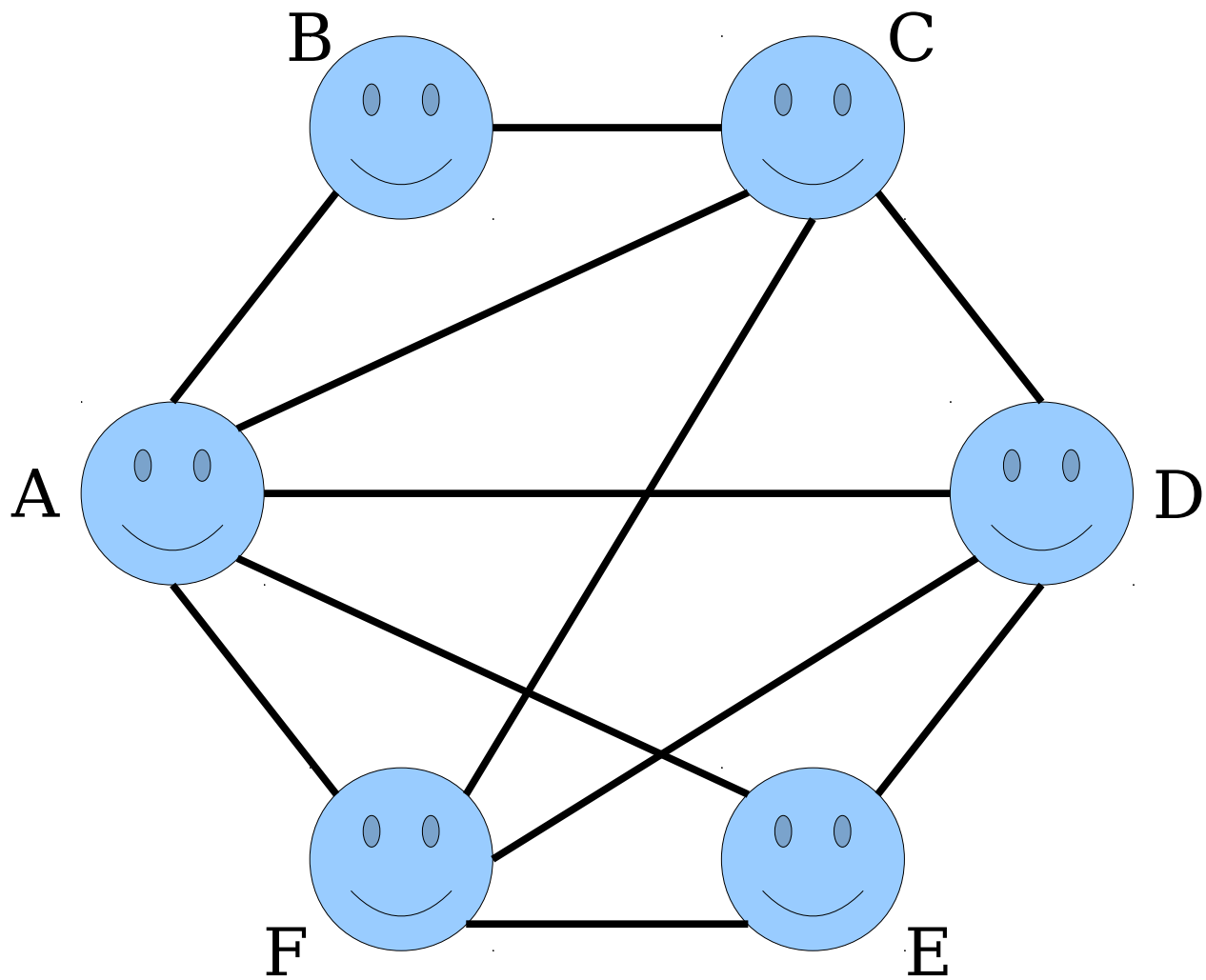


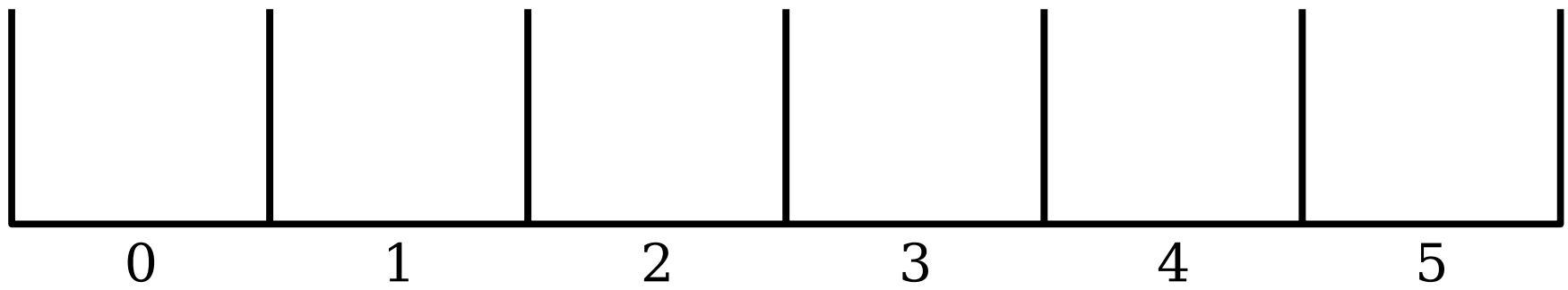
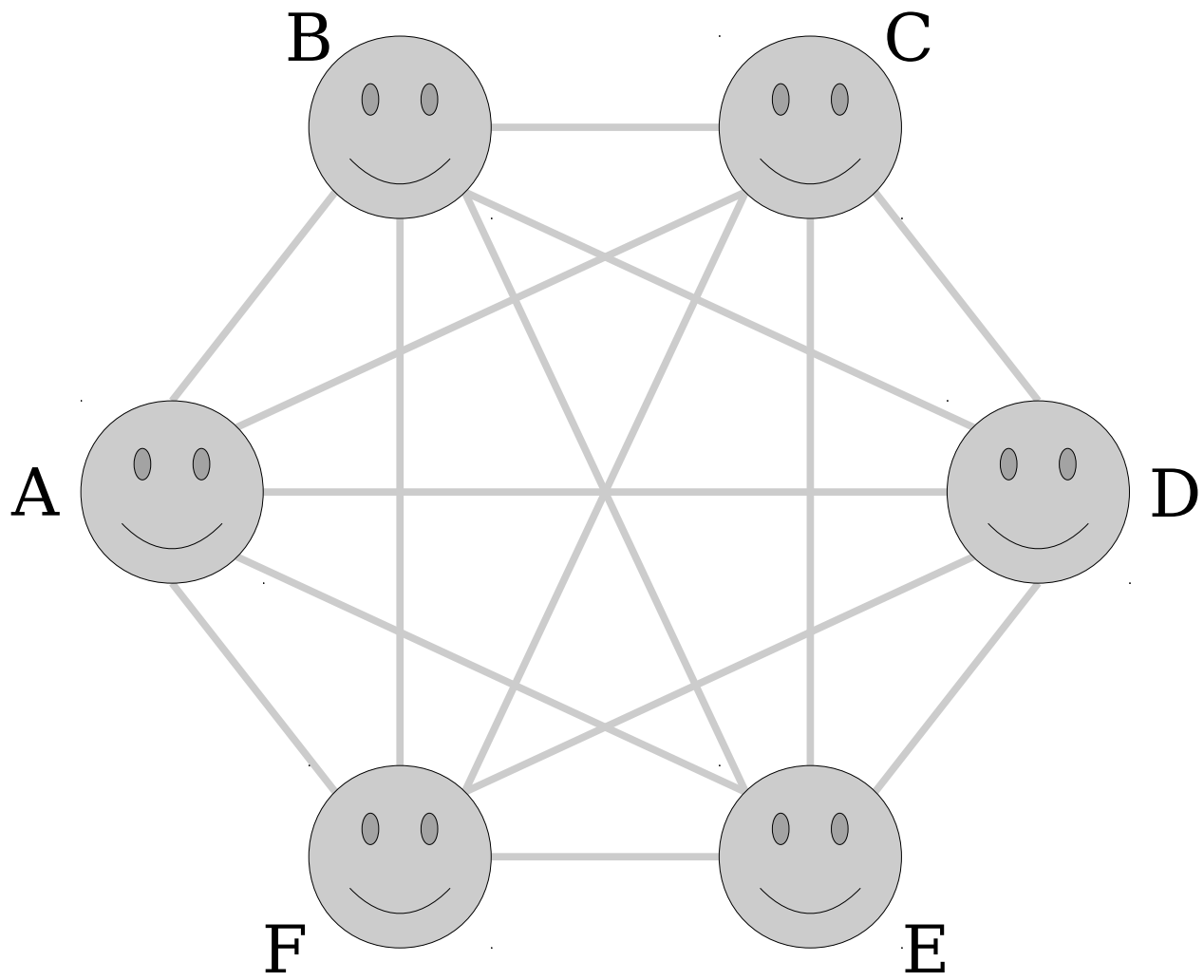
Degrees

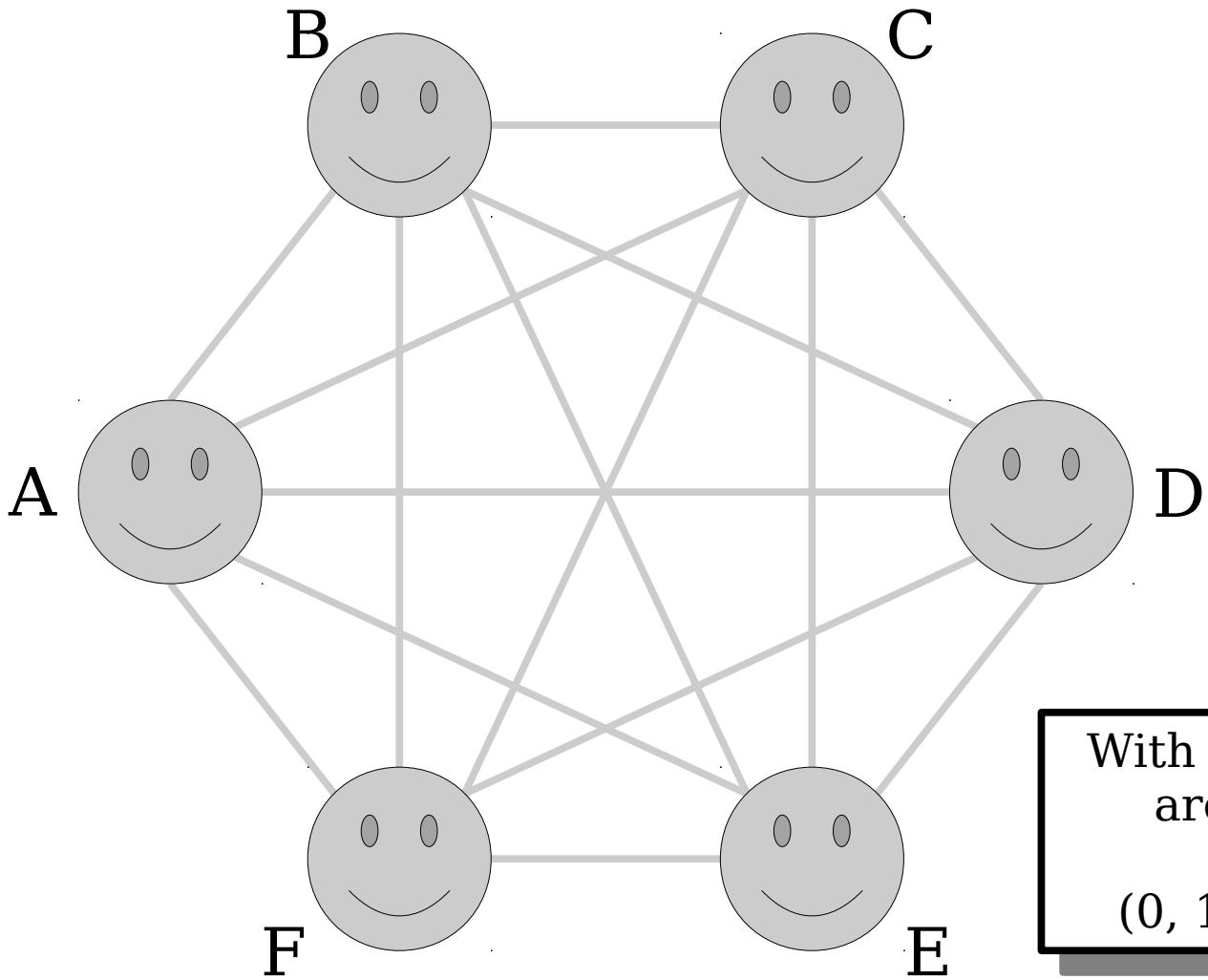
- The **degree** of a node v in a graph is the number of nodes that v is adjacent to.



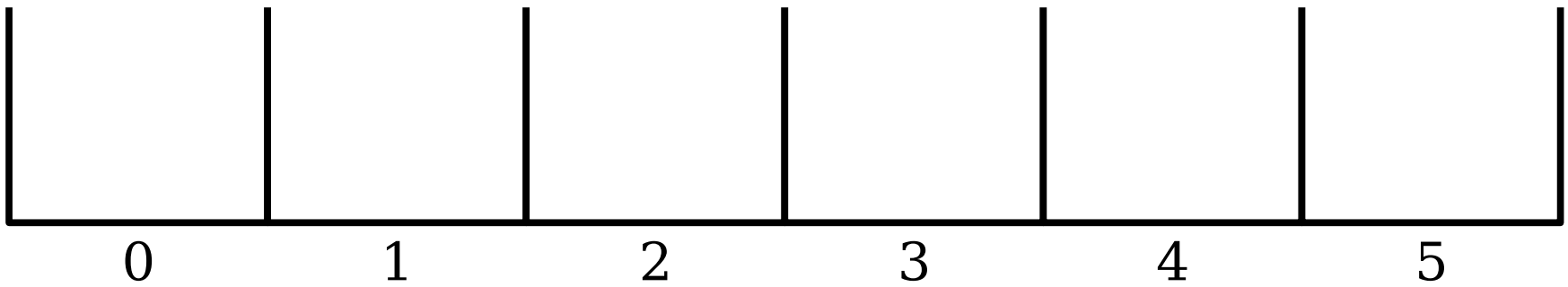
- Theorem:** Every graph with at least two nodes has at least two nodes with the same degree.
 - Equivalently: at any party with at least two people, there are at least two people with the same number of Facebook friends at the party.

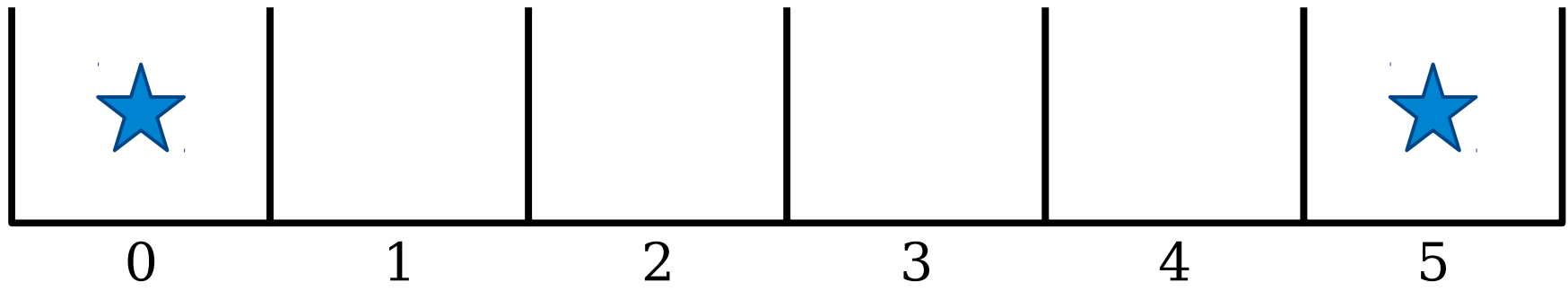
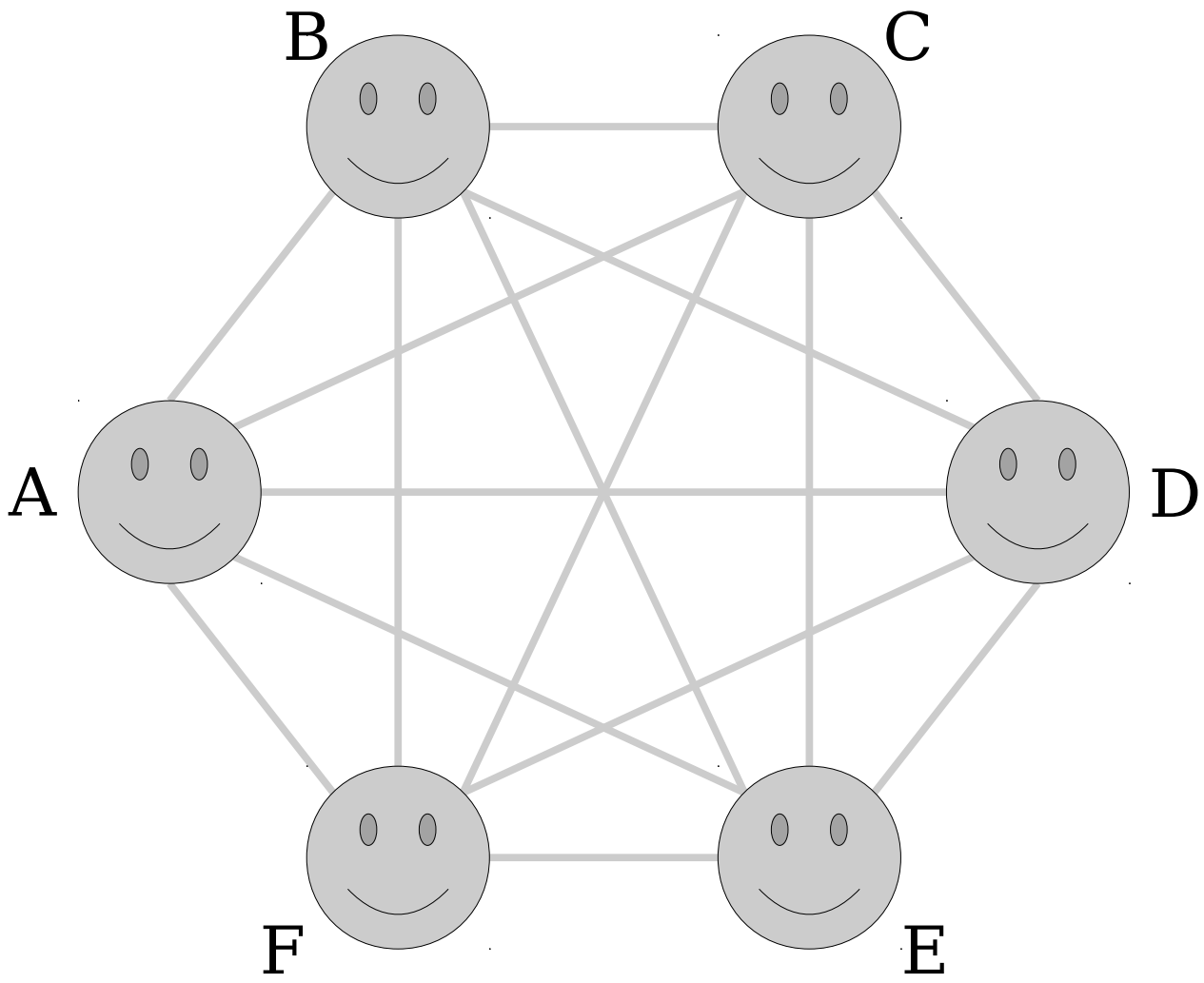


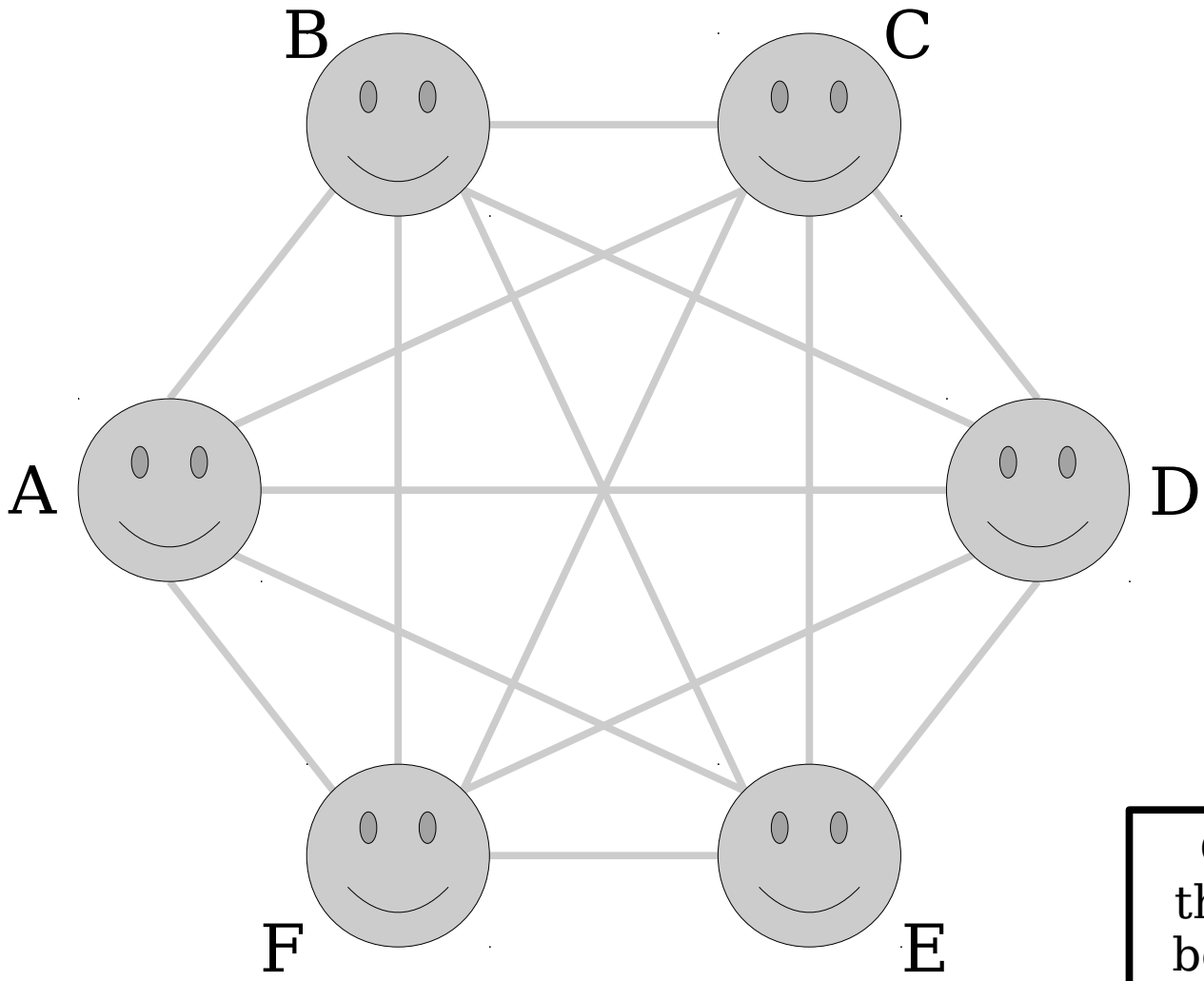




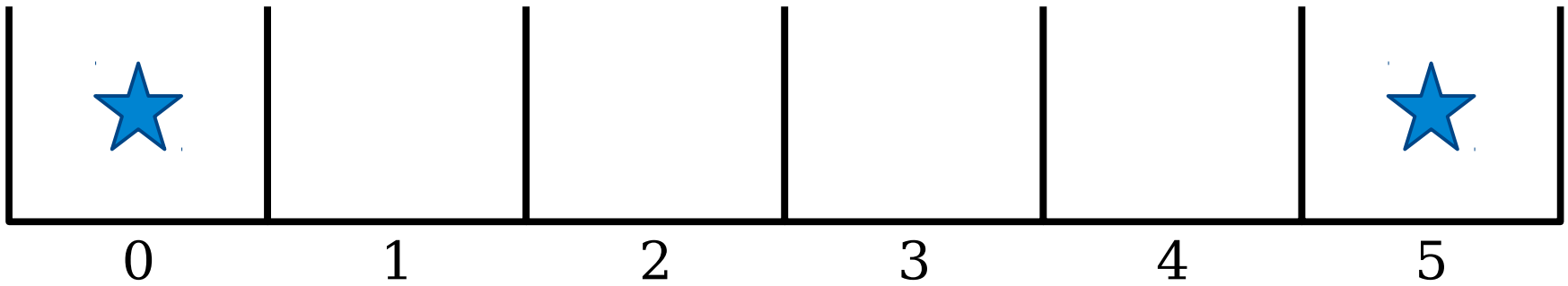
With n nodes, there are n possible degrees
(0, 1, 2, ..., $n - 1$)

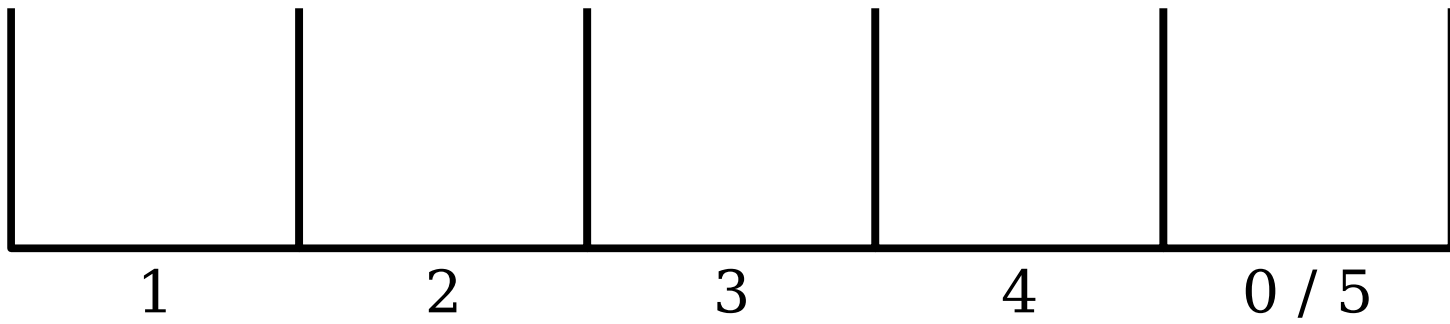
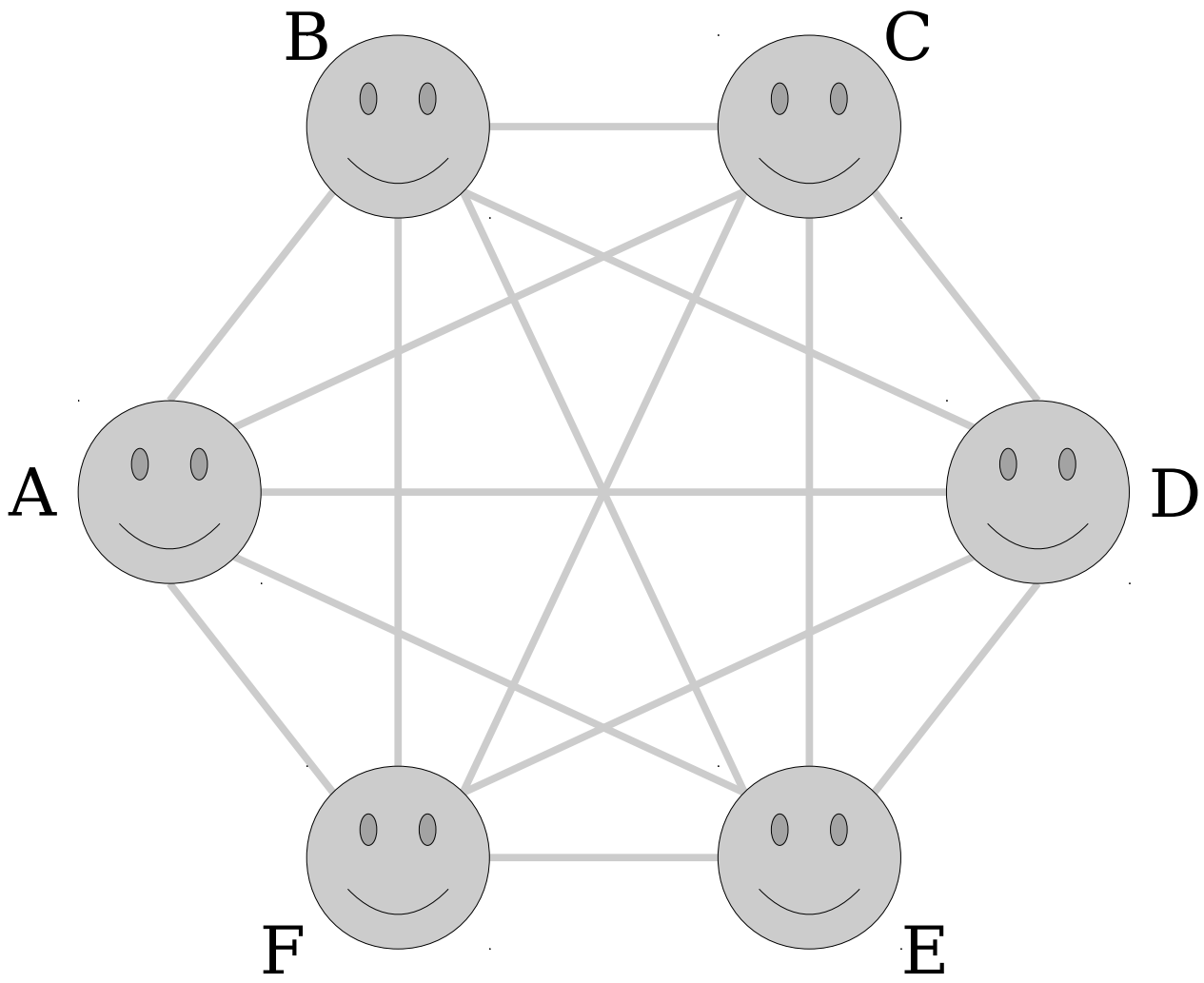






Can both of these buckets be nonempty?





Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

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Proof 1:

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We claim that G cannot simultaneously have a node u of degree 0 and a node v of degree $n - 1$:

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We therefore see that the possible options for degrees of nodes in G are either drawn from $0, 1, \dots, n - 2$ or from $1, 2, \dots, n - 1$.

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Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G , namely $0, 1, 2, \dots, n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

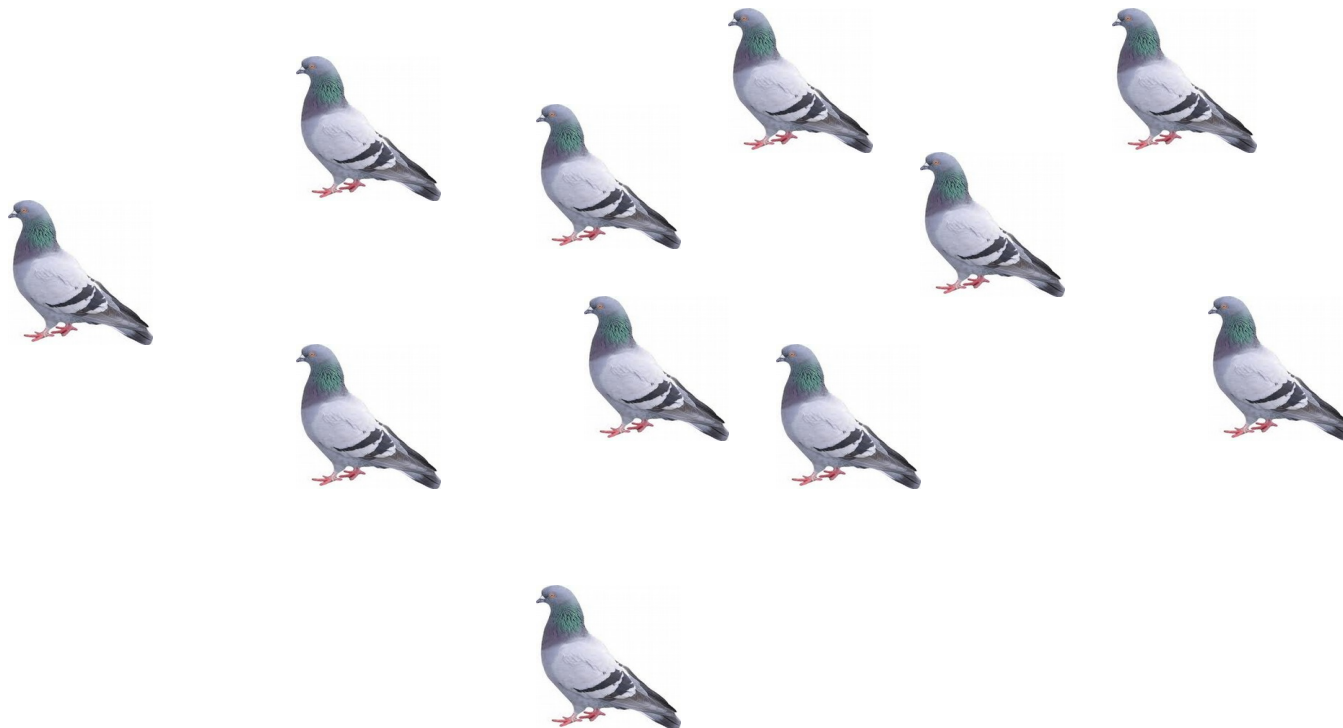
Suppose 11 objects are distributed into 5 bins.
How many of the following statements are true?

- U*: The bin with the most objects must contain at least 2 objects.
- V*: The bin with the most objects must contain at least 3 objects.
- W*: The bin with the most objects must contain at least 4 objects.
- X*: The bin with the fewest objects must contain at most 1 object.
- Y*: The bin with the fewest objects must contain at most 2 objects.
- Z*: The bin with the fewest objects must contain at most 3 objects.

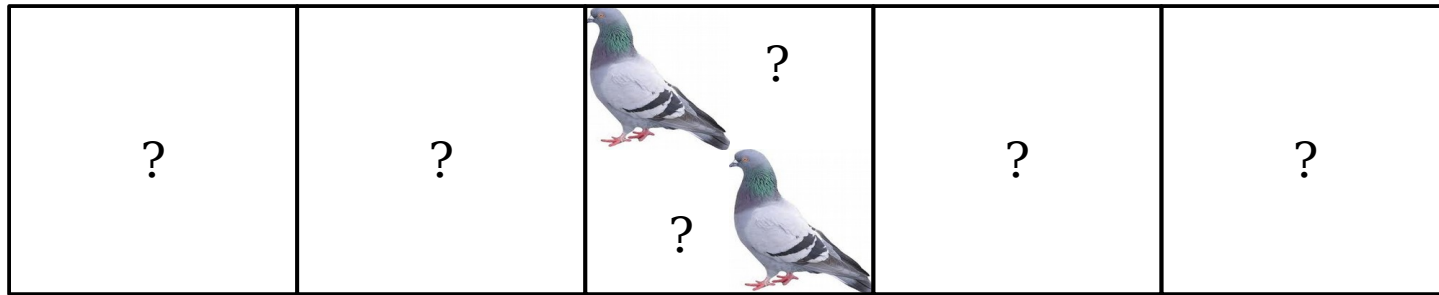
(If there are many bins tied for the most or fewest objects, you can pick any one of them)

Answer at **PolleEv.com/cs103** or
text **CS103** to **22333** once to join, then a number.

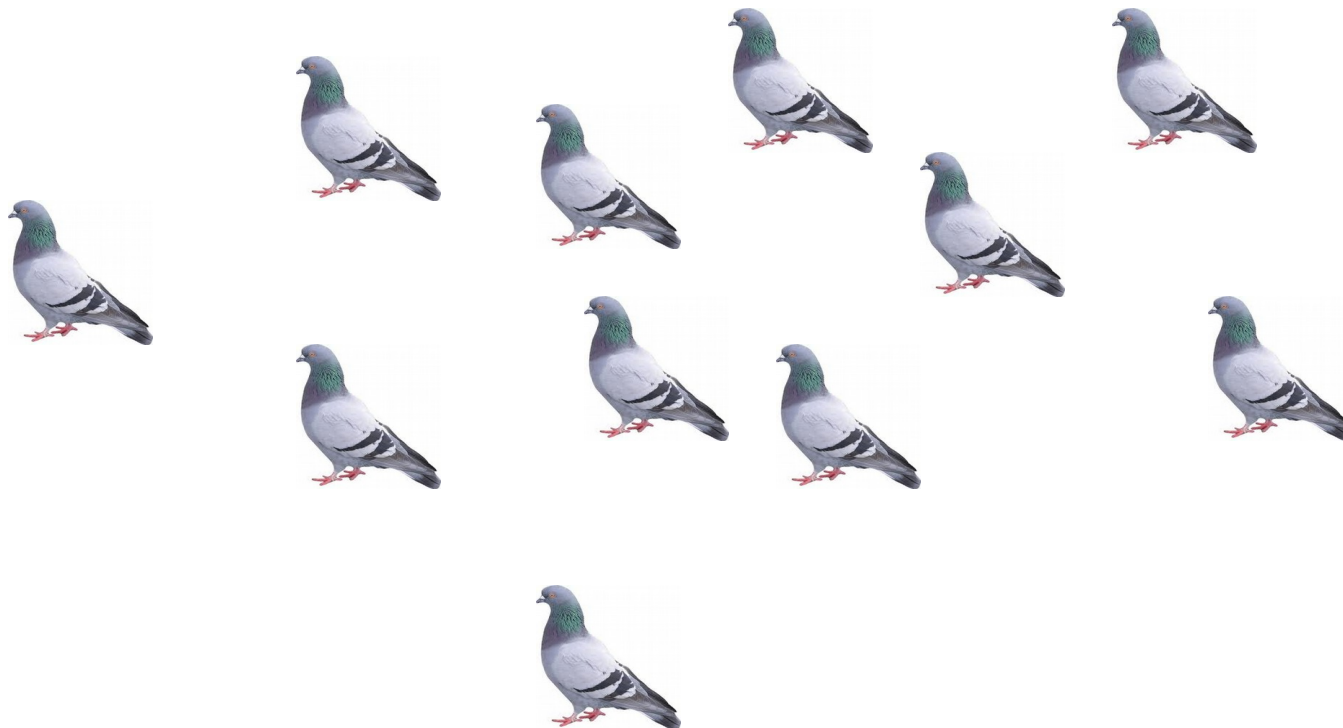
The Pigeonhole Principle



The Pigeonhole Principle



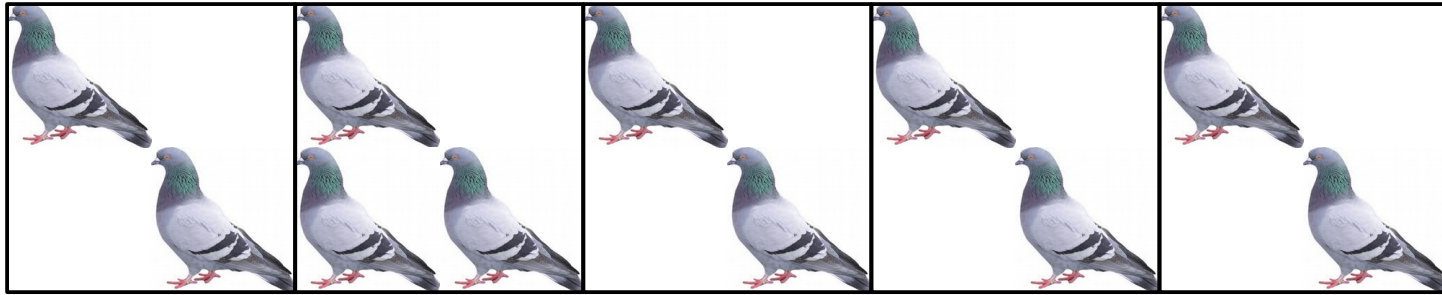
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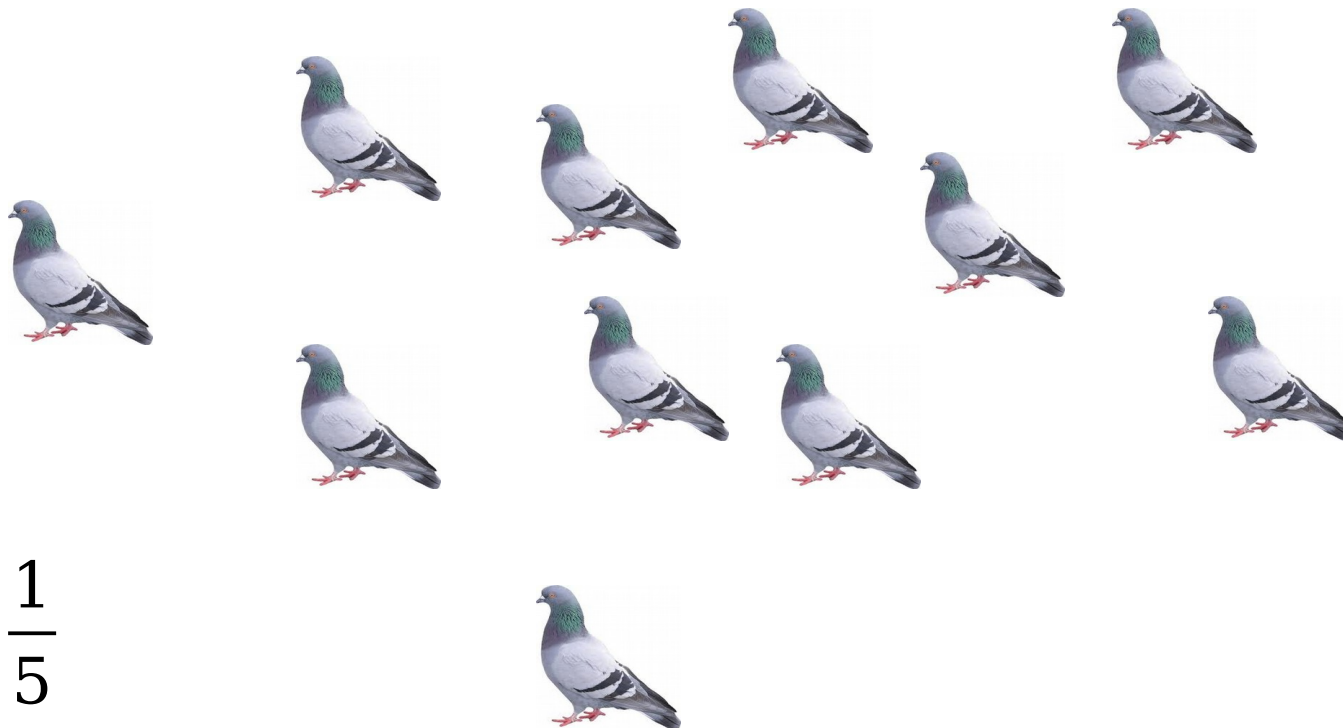
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The Pigeonhole Principle

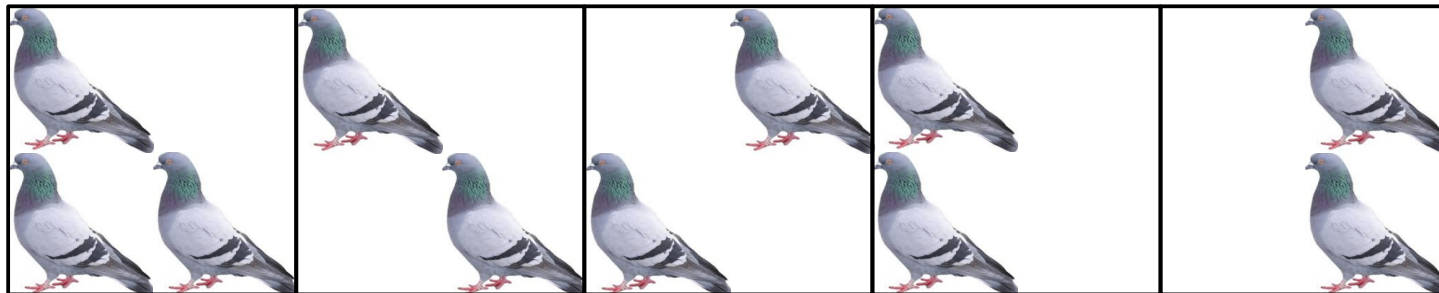


$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The **generalized pigeonhole principle** says that if you distribute m objects into n bins, then
 - some bin will have at least $\lceil m/n \rceil$ objects in it, and
 - some bin will have at most $\lfloor m/n \rfloor$ objects in it.

$\lceil m/n \rceil$ means “ m/n , rounded up.”
 $\lfloor m/n \rfloor$ means “ m/n , rounded down.”



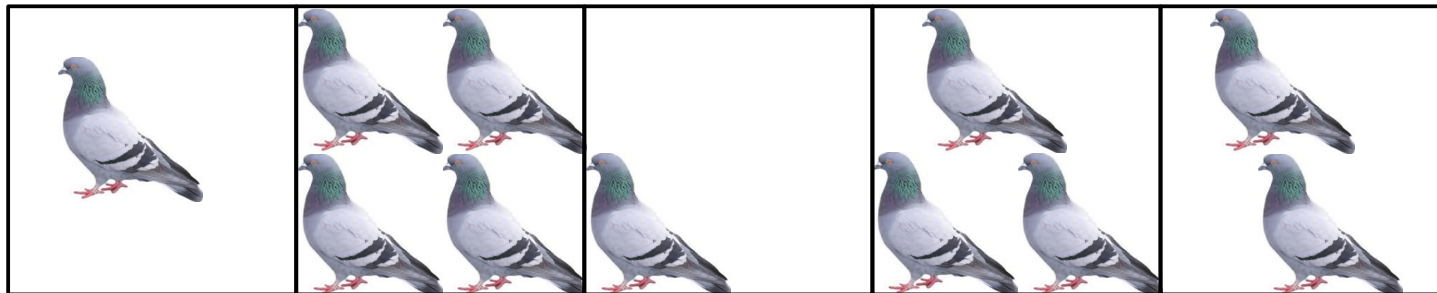
$$m = 11$$
$$n = 5$$

$$\lceil m / n \rceil = 3$$
$$\lfloor m / n \rfloor = 2$$

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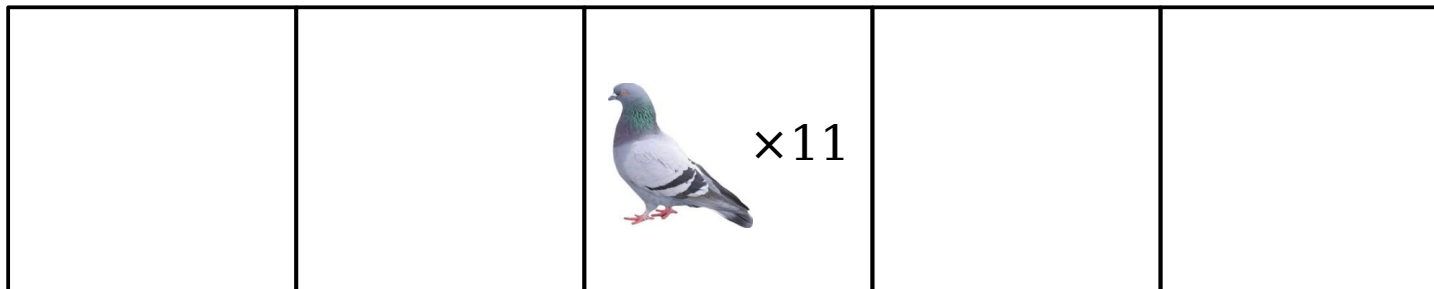
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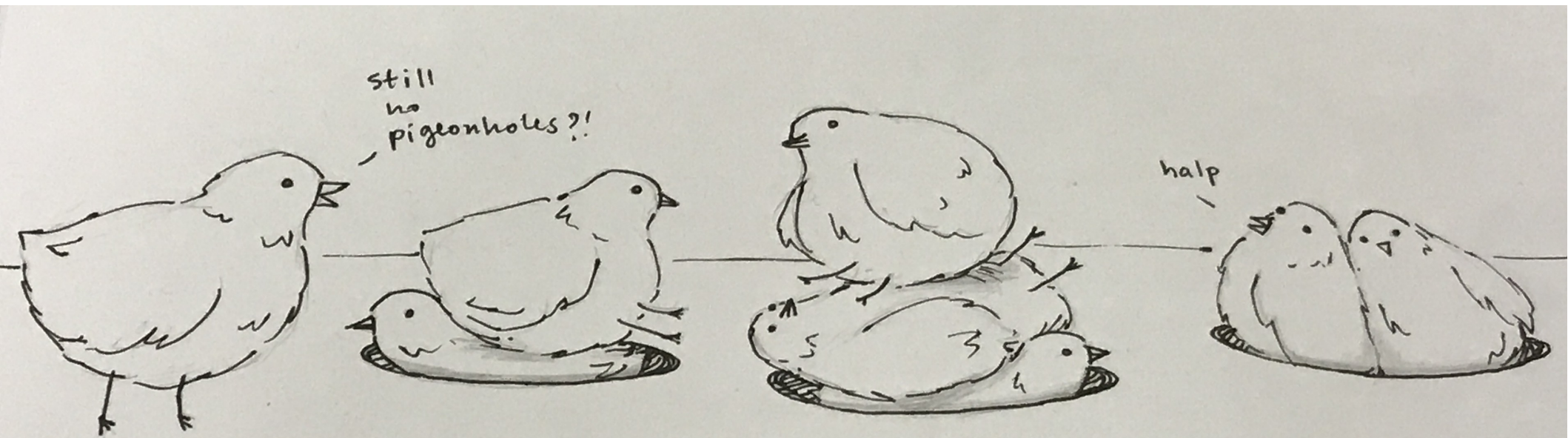
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$$m = 11$$
$$n = 5$$

$$\lceil m / n \rceil = 3$$
$$\lfloor m / n \rfloor = 2$$



$$m = 8, n = 3$$

Thanks to Amy Liu for this awesome drawing!

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n , there is a way to distribute m objects into n bins such that each bin contains fewer than $\lceil m/n \rceil$ objects.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . Since there are m objects in total, we know that

$$m = x_1 + x_2 + \dots + x_n.$$

Since each bin contains fewer than $\lceil m/n \rceil$ objects, we see that $x_i < \lceil m/n \rceil$ for each i . Therefore, we have that

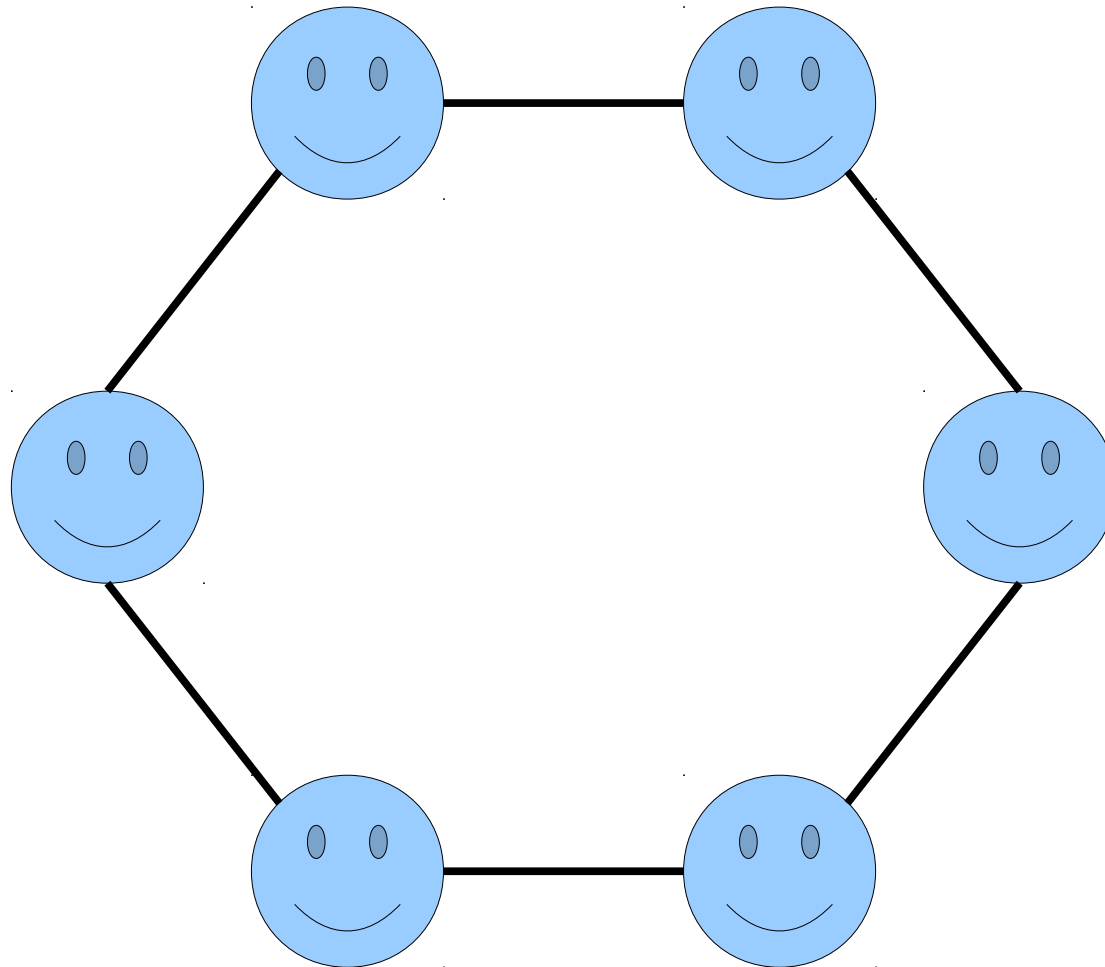
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &< \lceil m/n \rceil + \lceil m/n \rceil + \dots + \lceil m/n \rceil \quad (n \text{ times}) \\ &= m. \end{aligned}$$

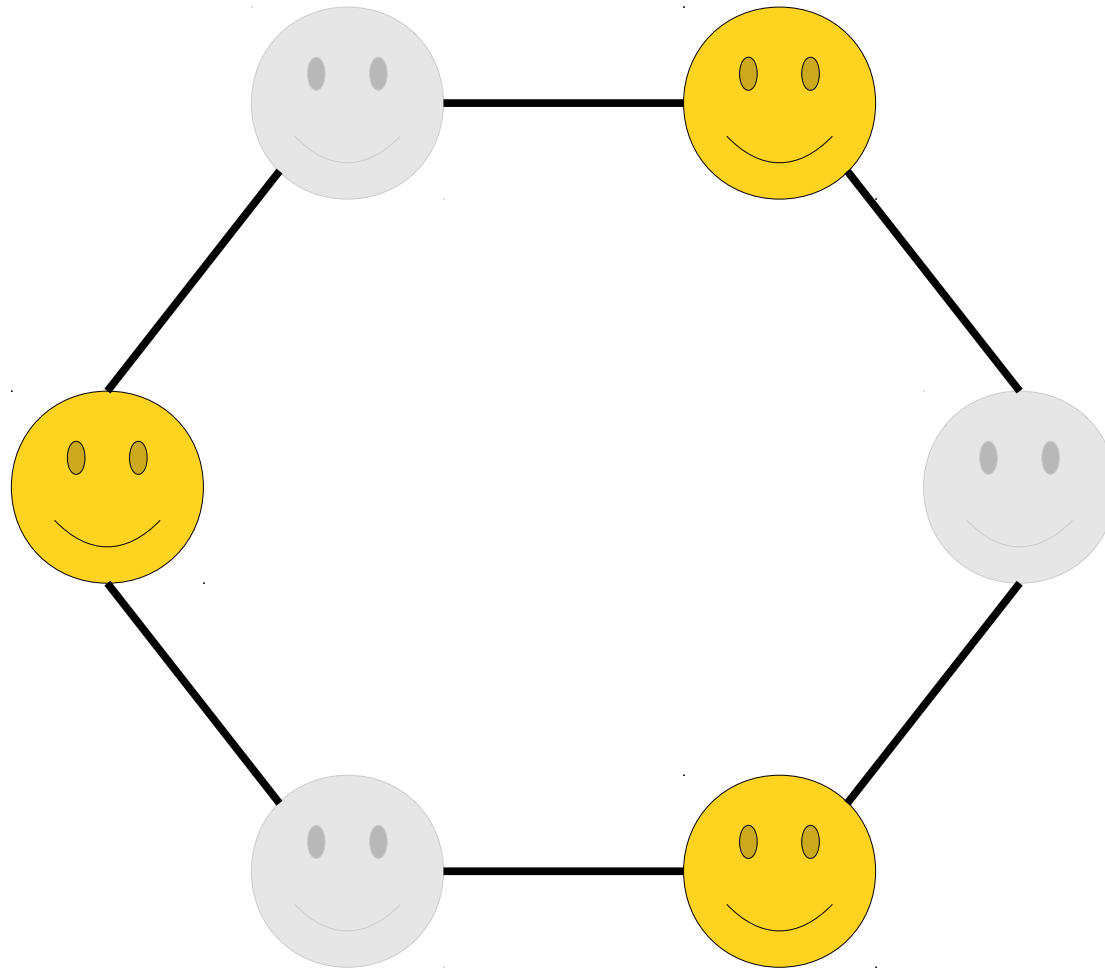
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

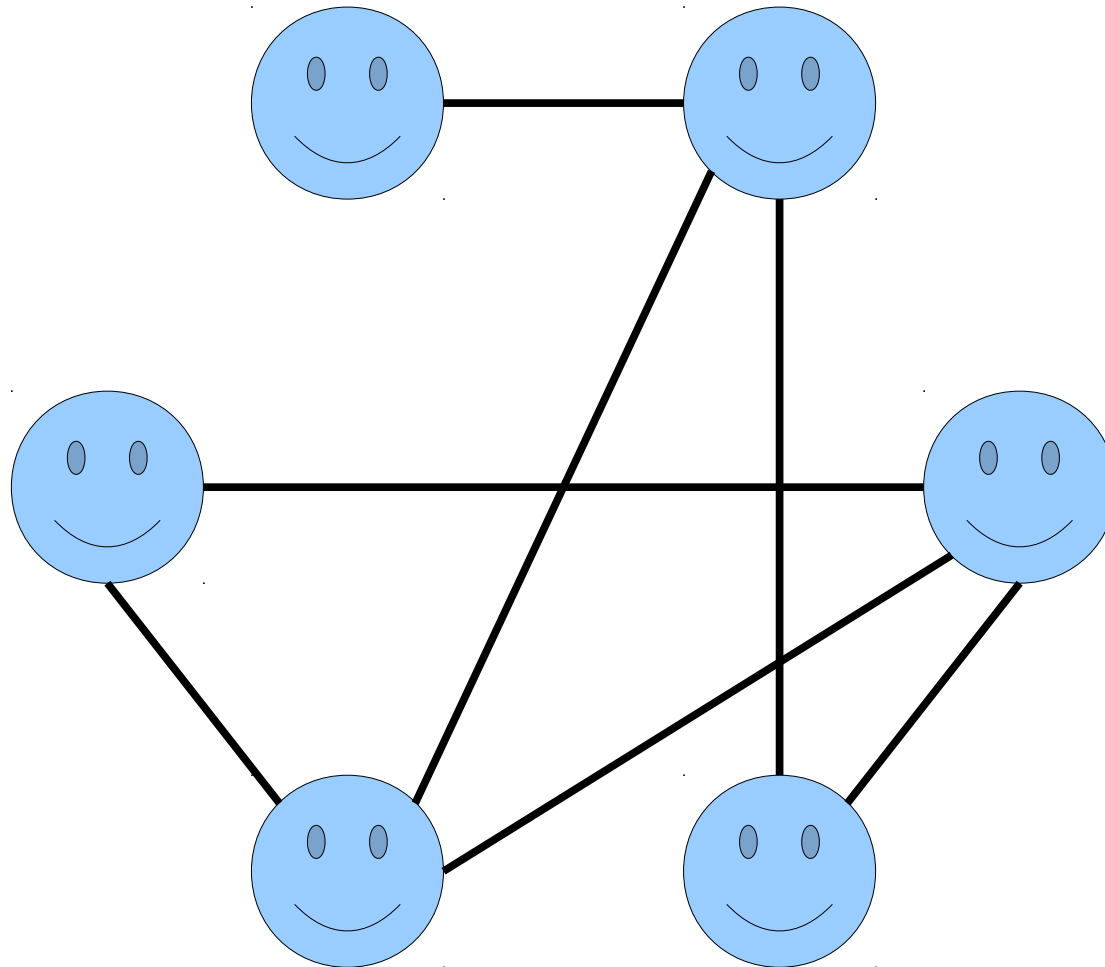
An Application: Friends and Strangers

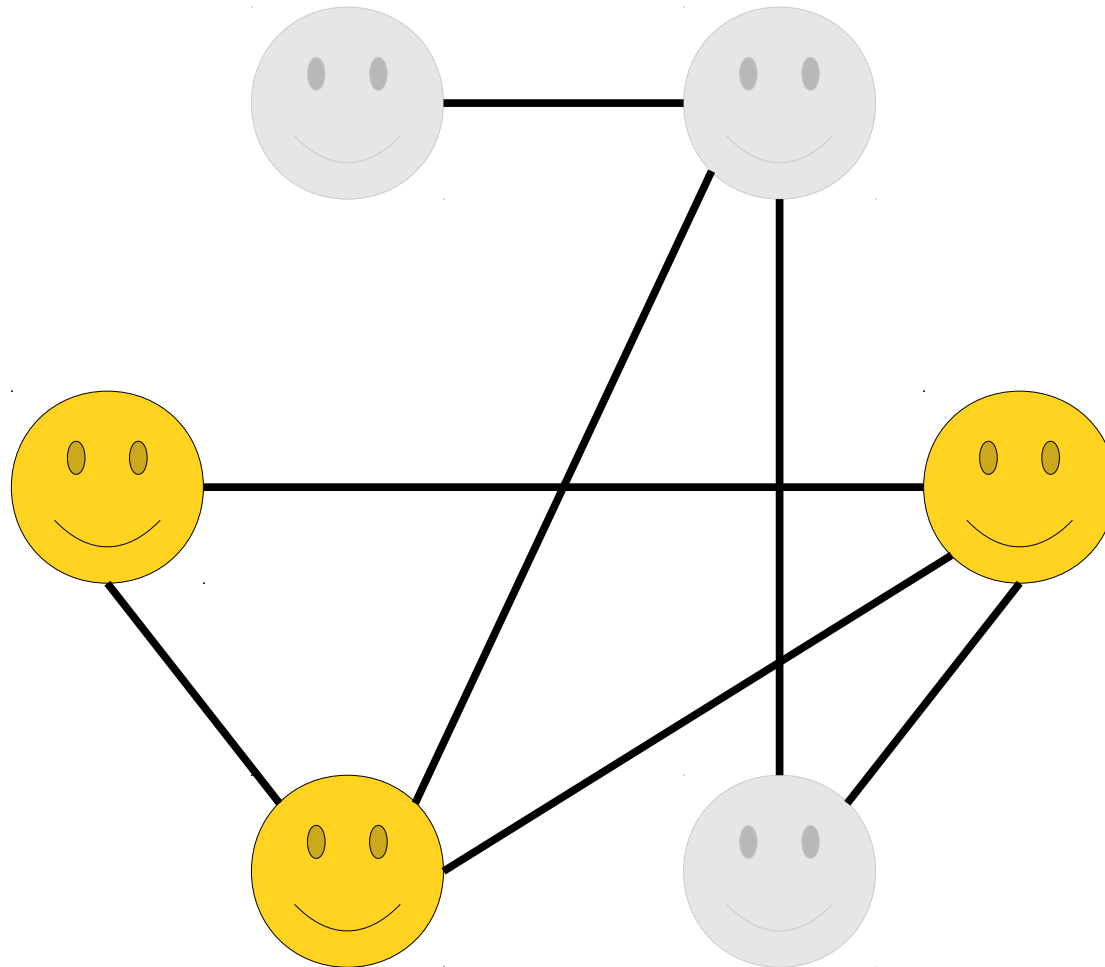
Friends and Strangers

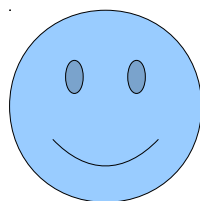
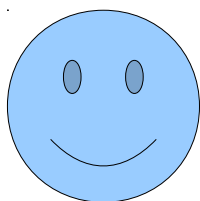
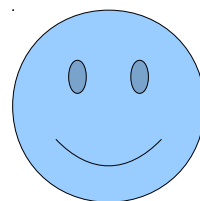
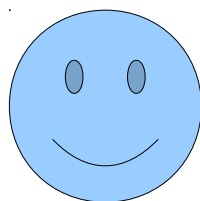
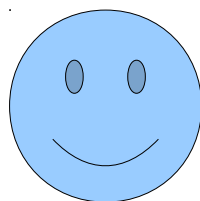
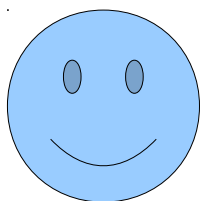
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

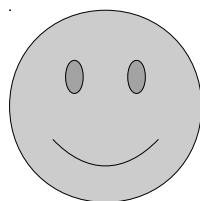
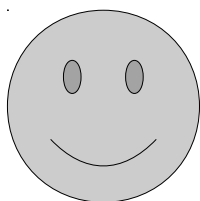
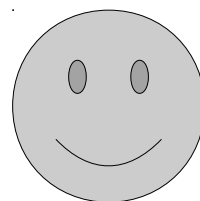
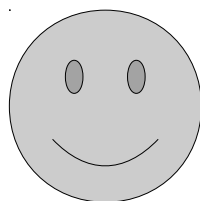
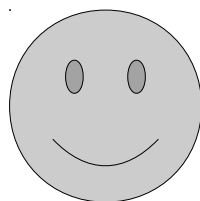
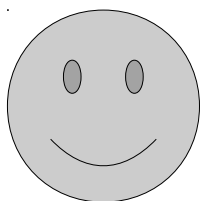


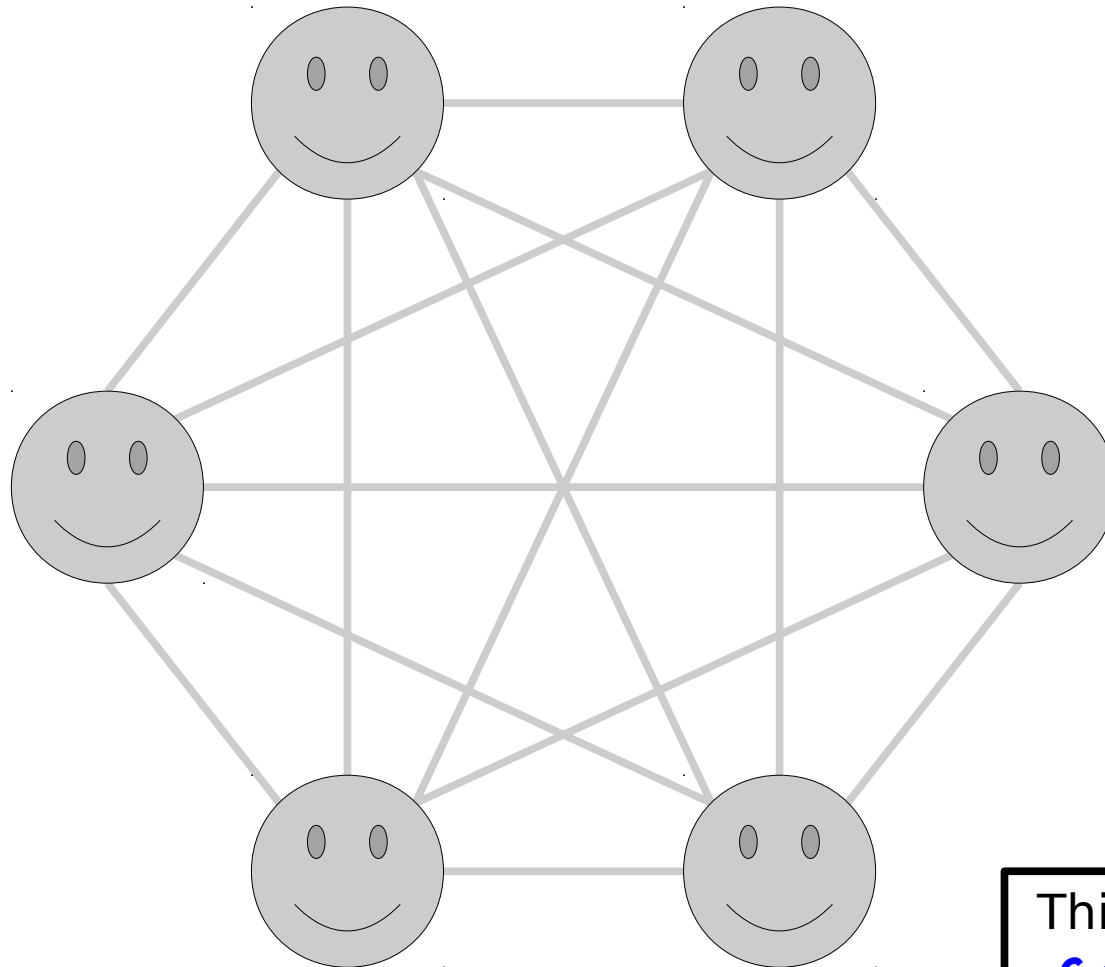




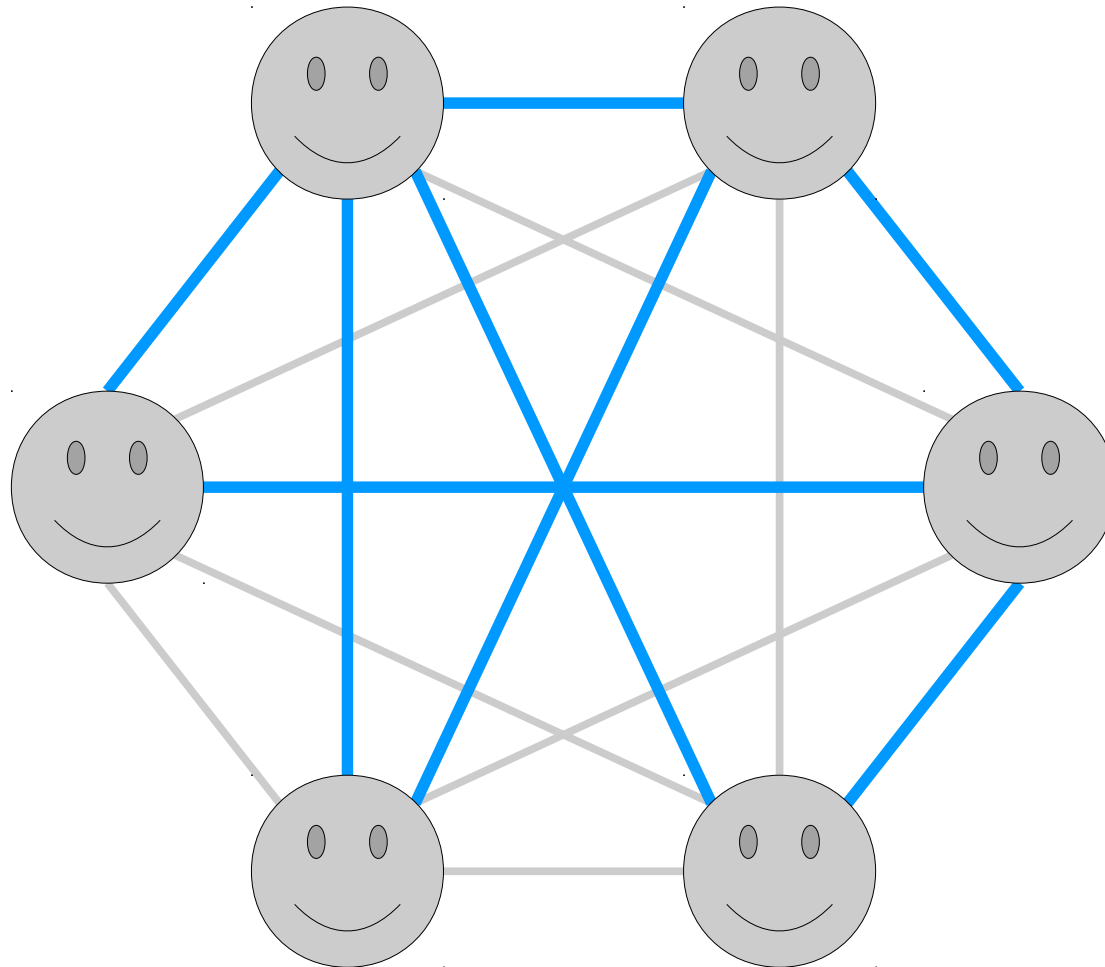


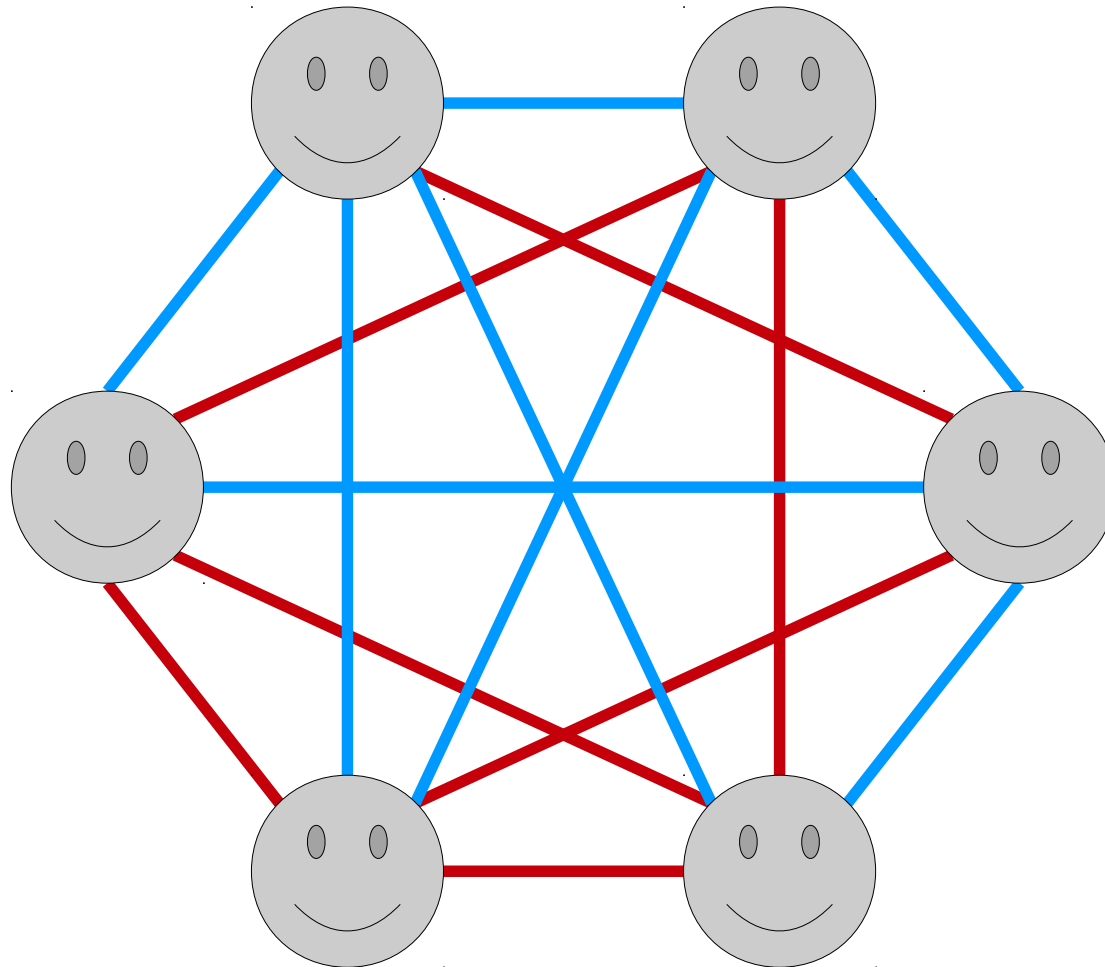


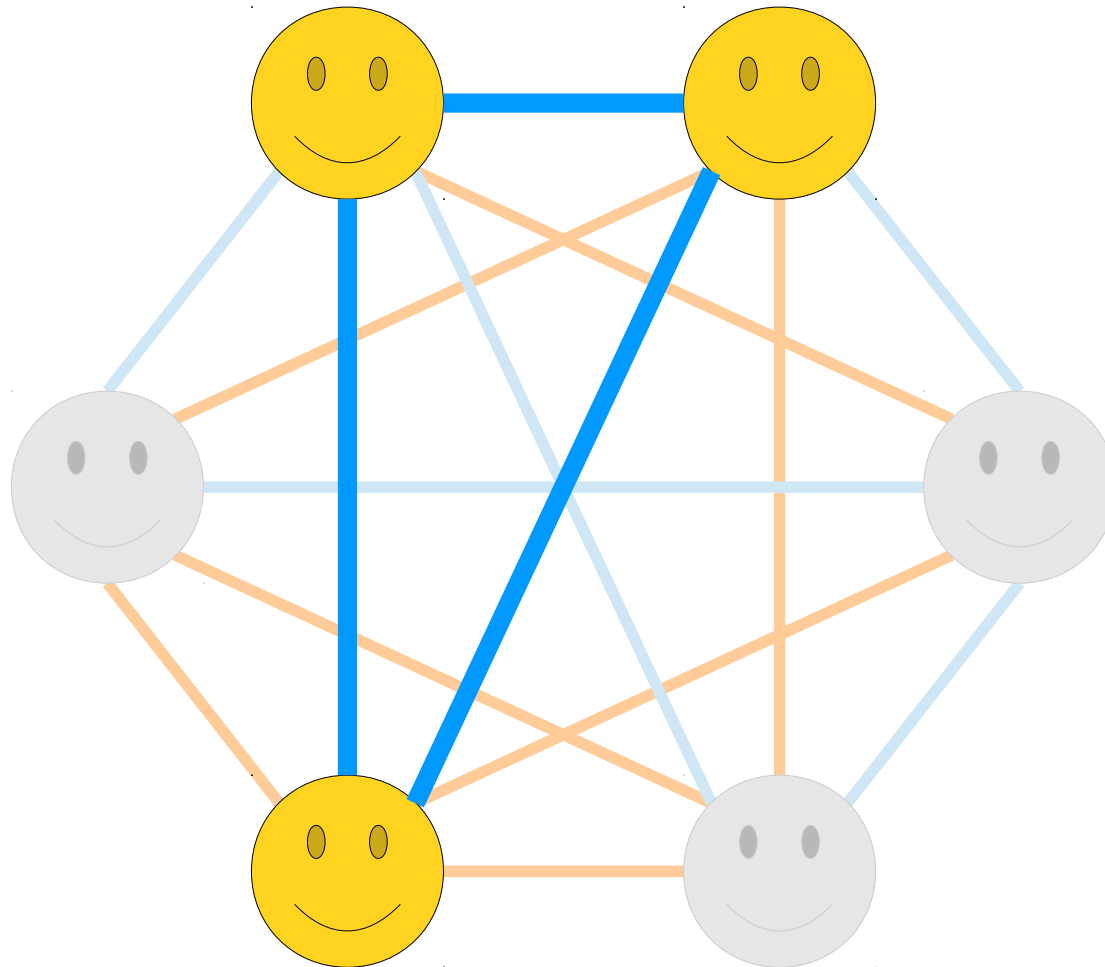


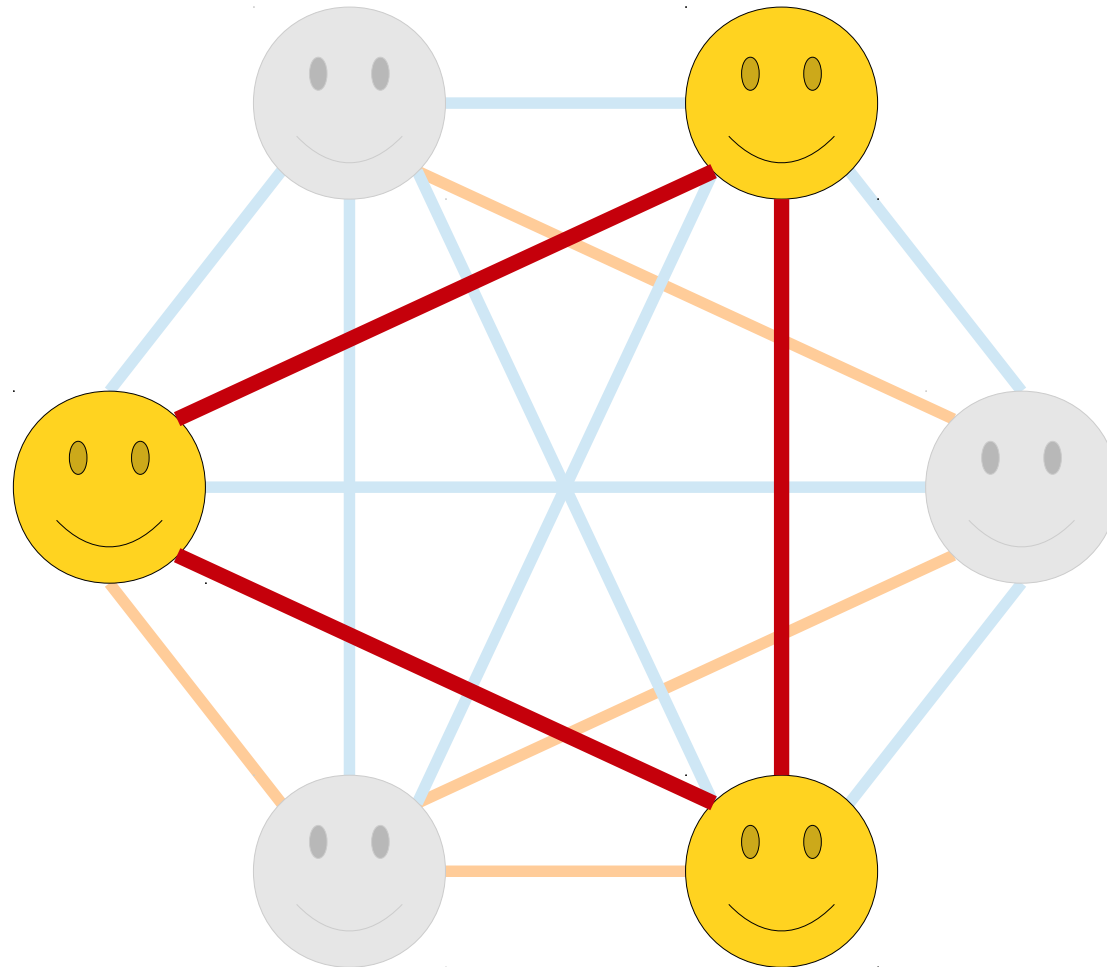


This graph is called a **6-clique**, by the way.



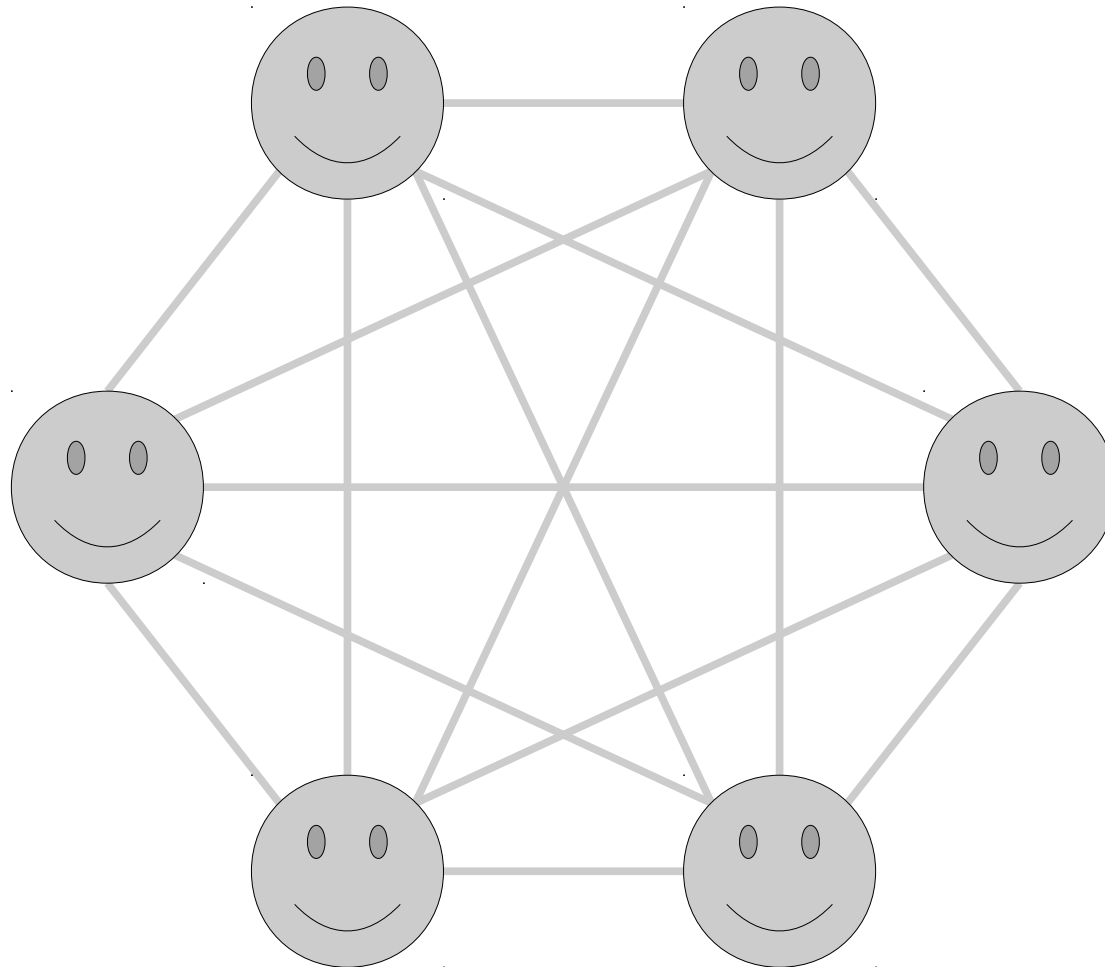


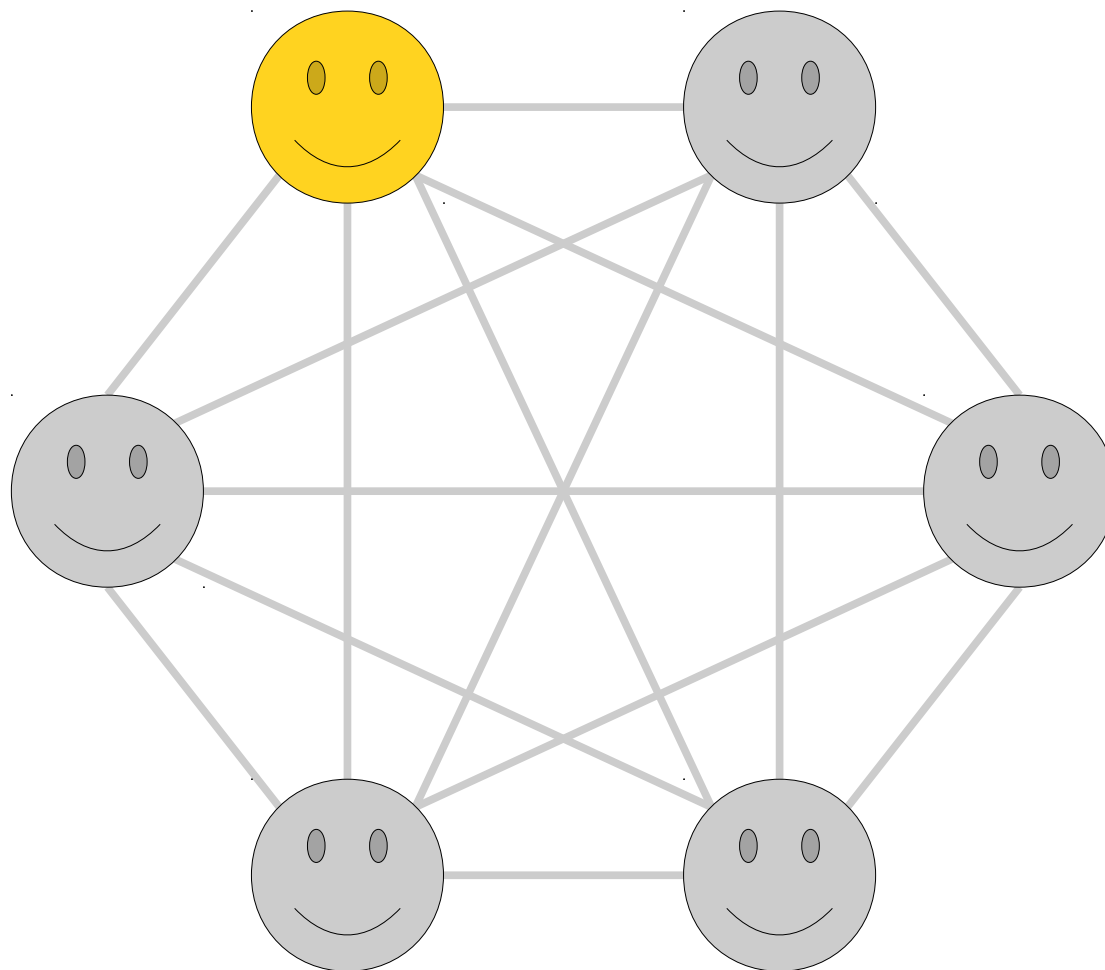


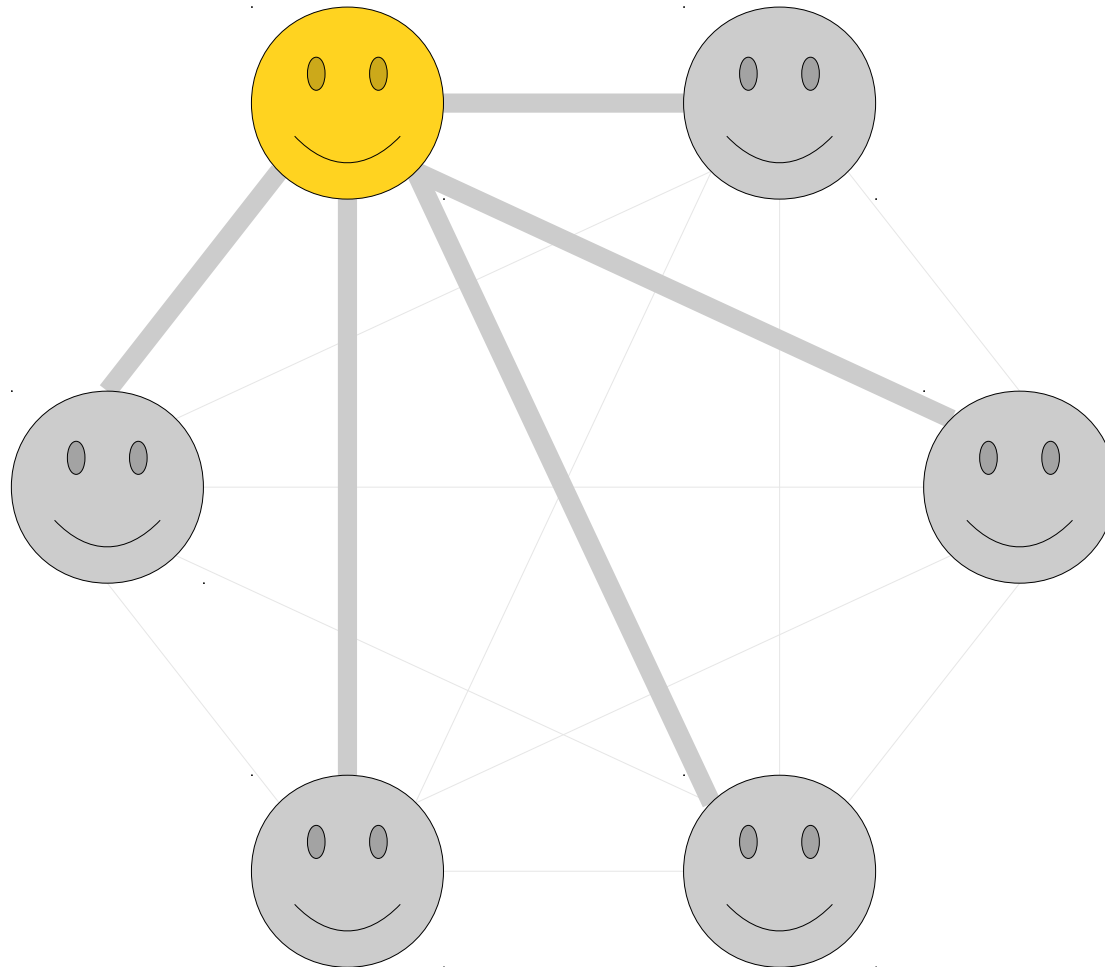


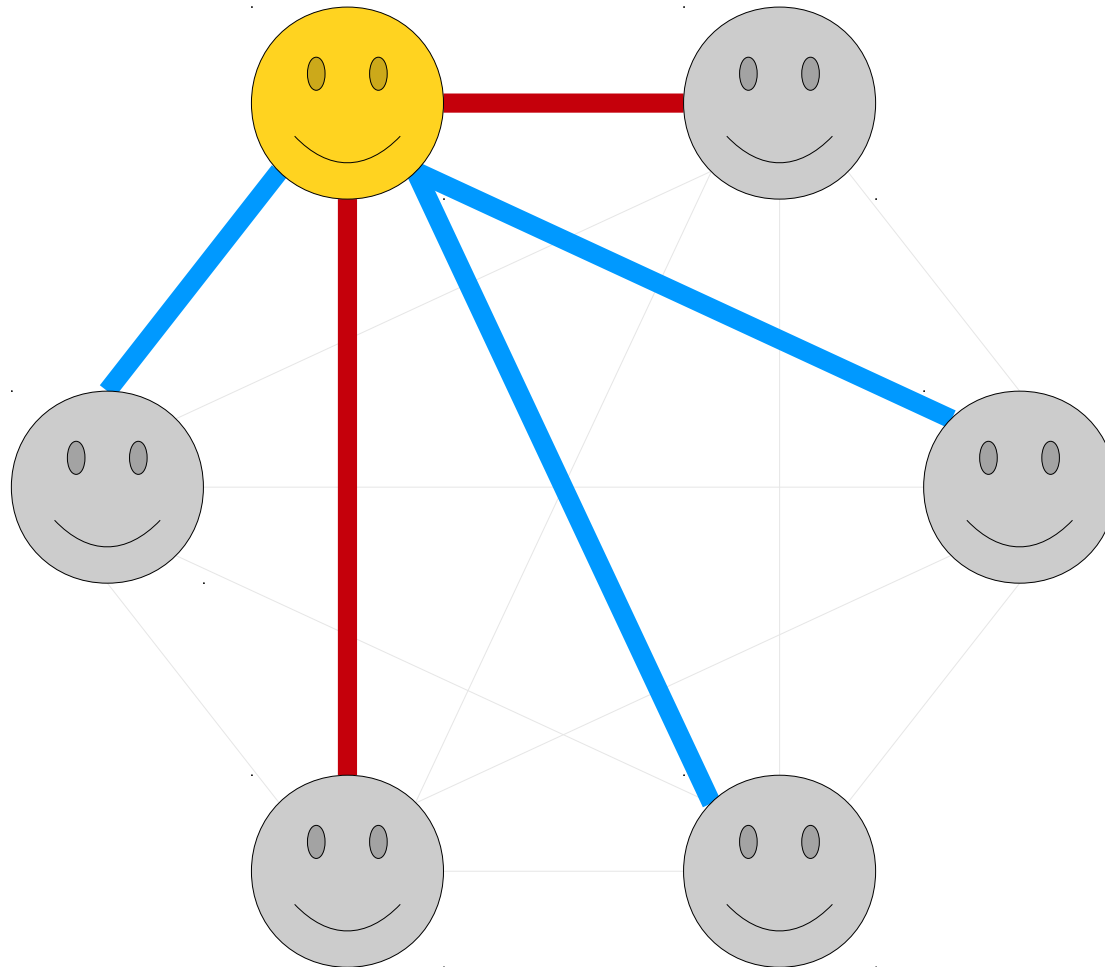
Friends and Strangers Restated

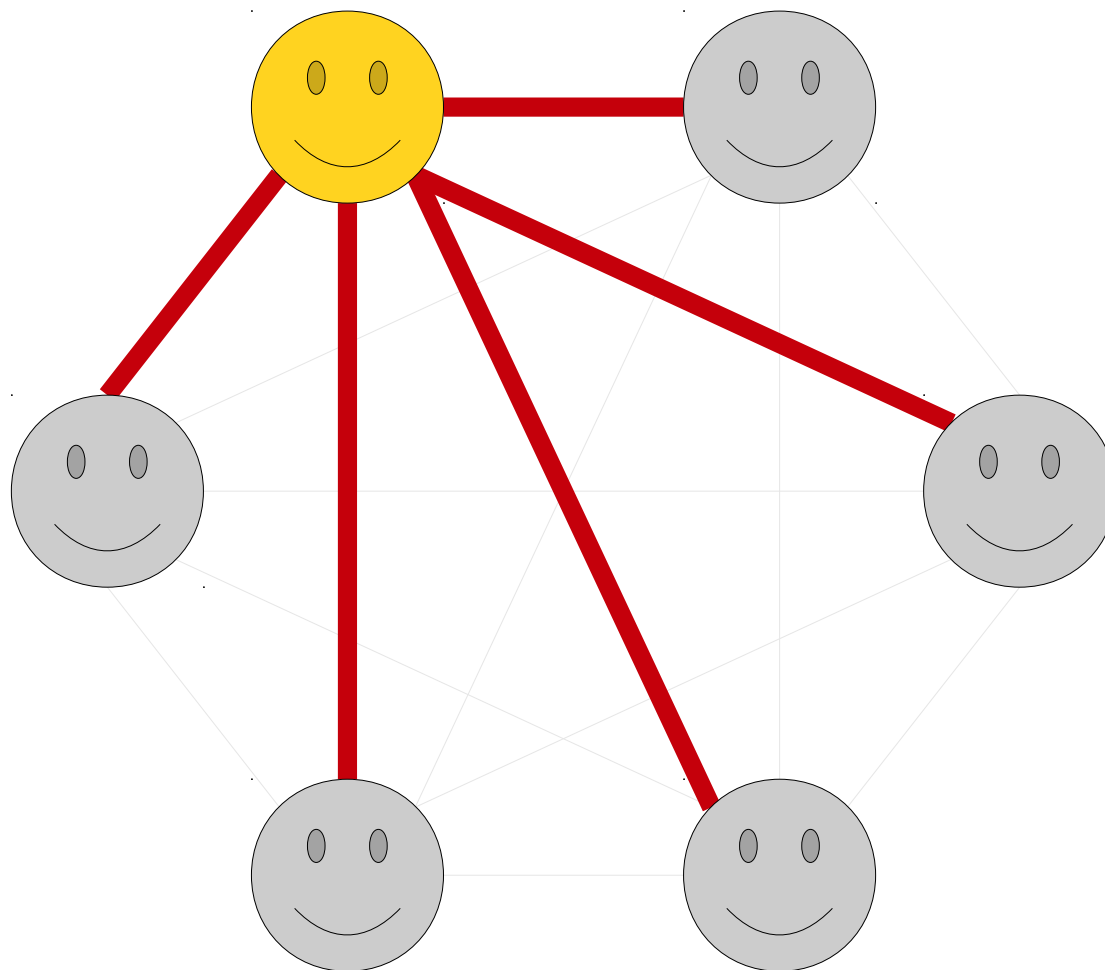
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:
- ***Theorem:*** Consider a 6-clique where every edge is colored red or blue. The graph contains a red triangle, a blue triangle, or both.
- How can we prove this?

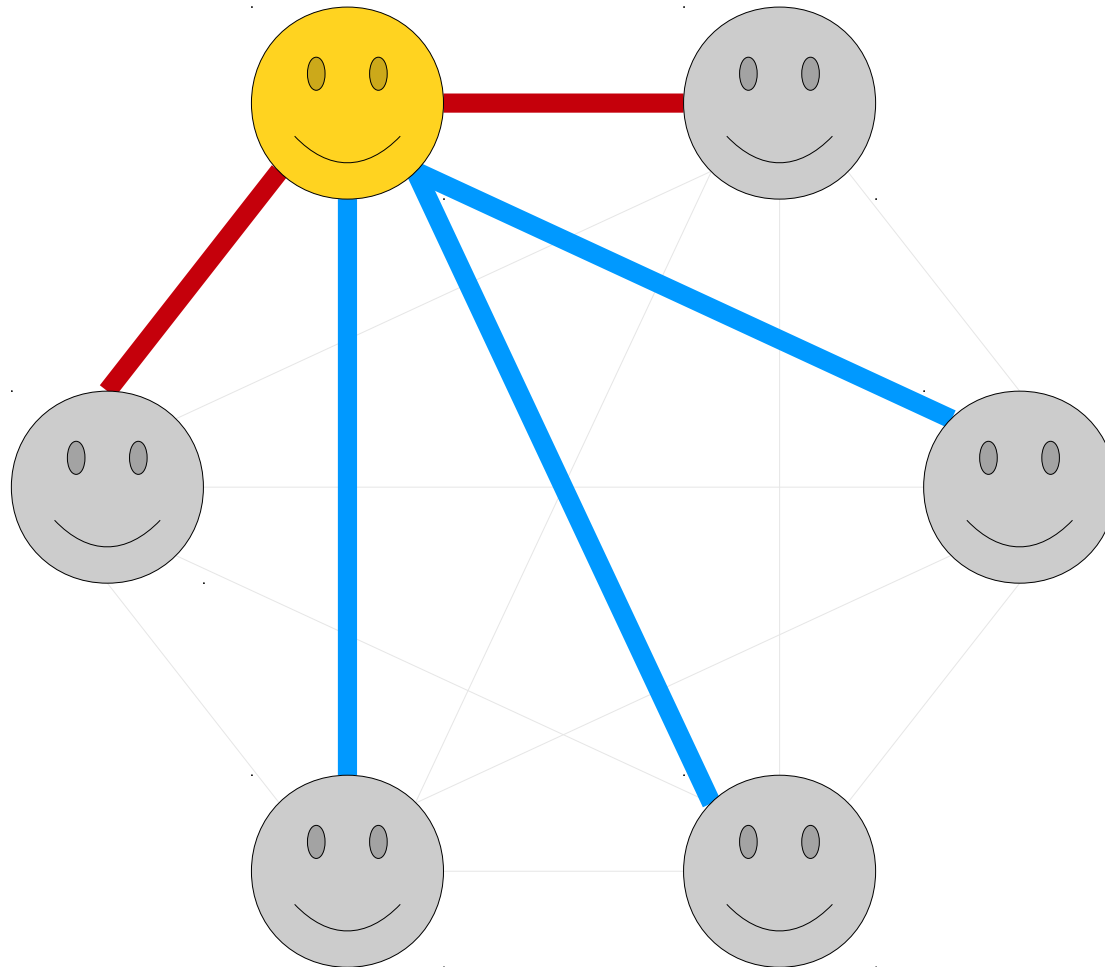


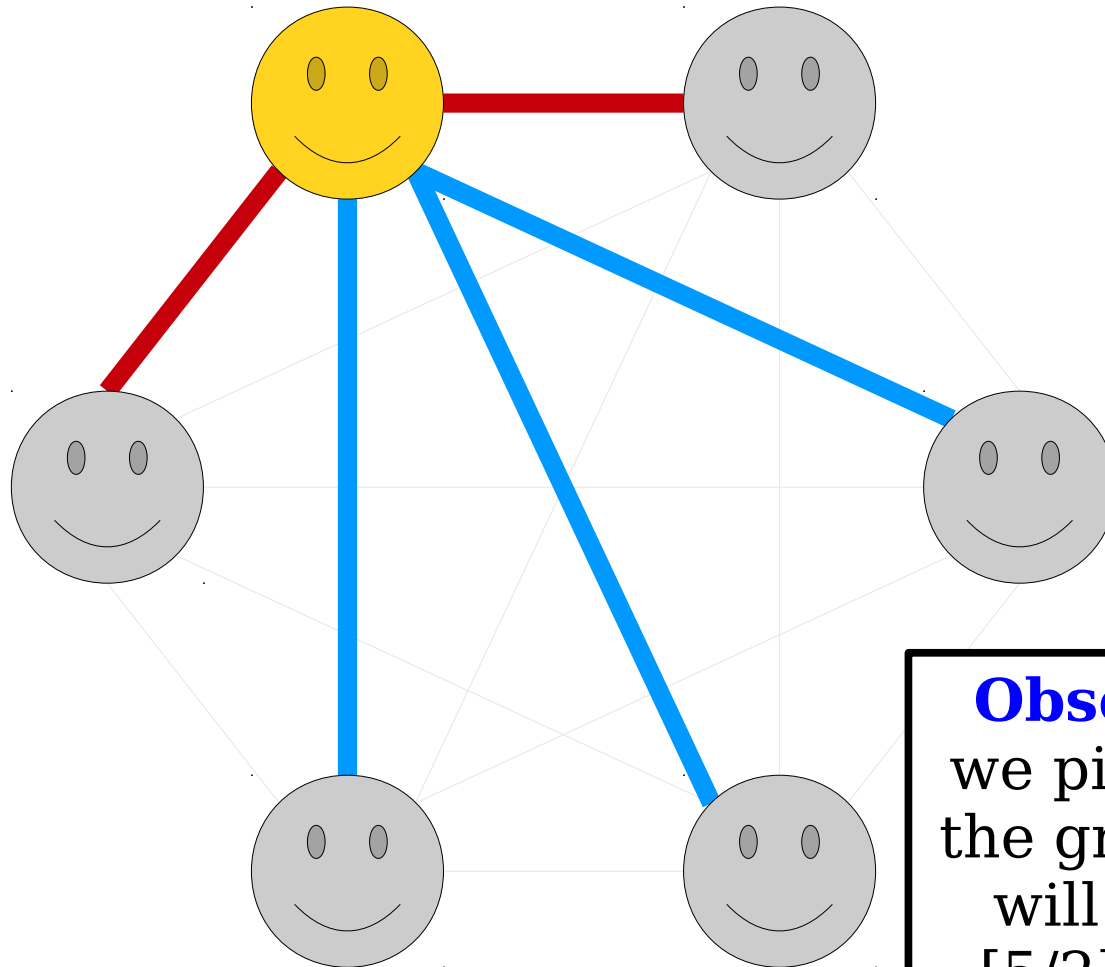




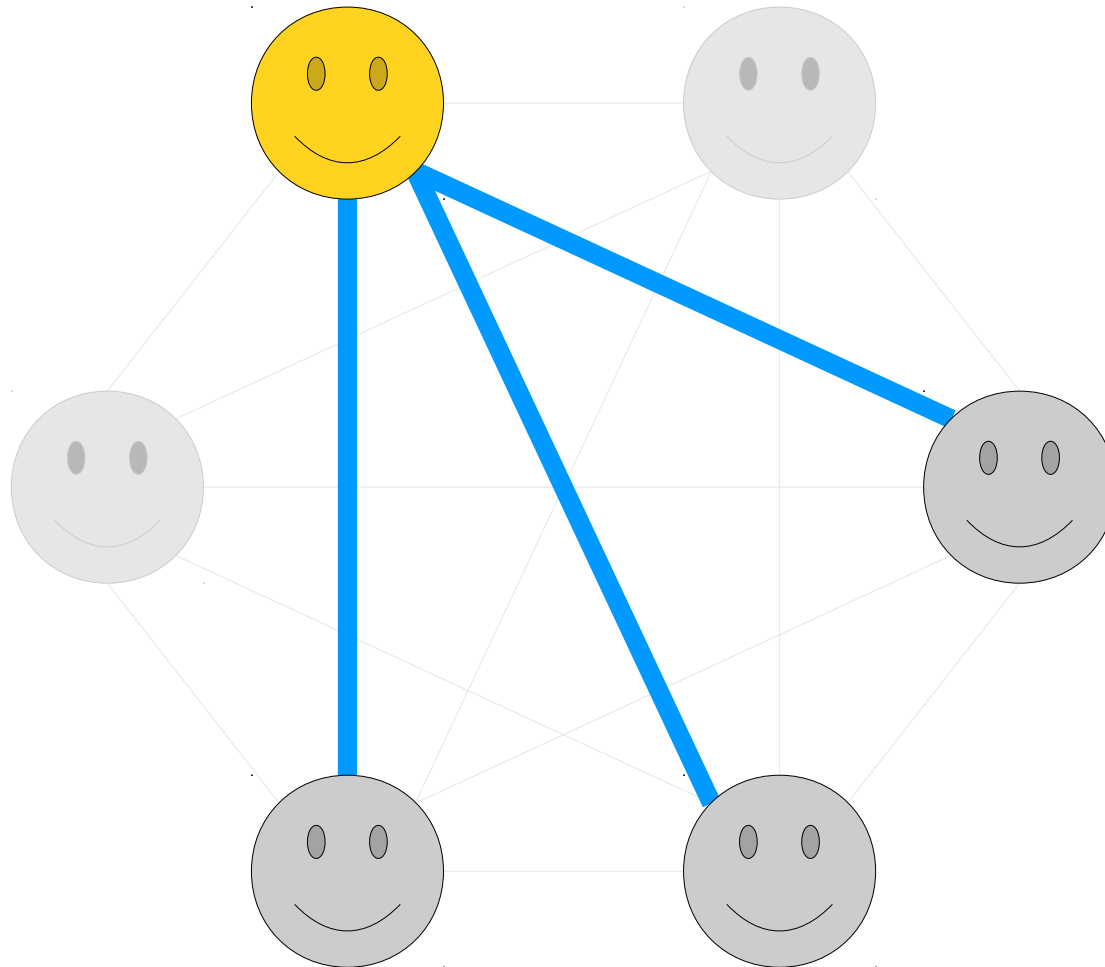


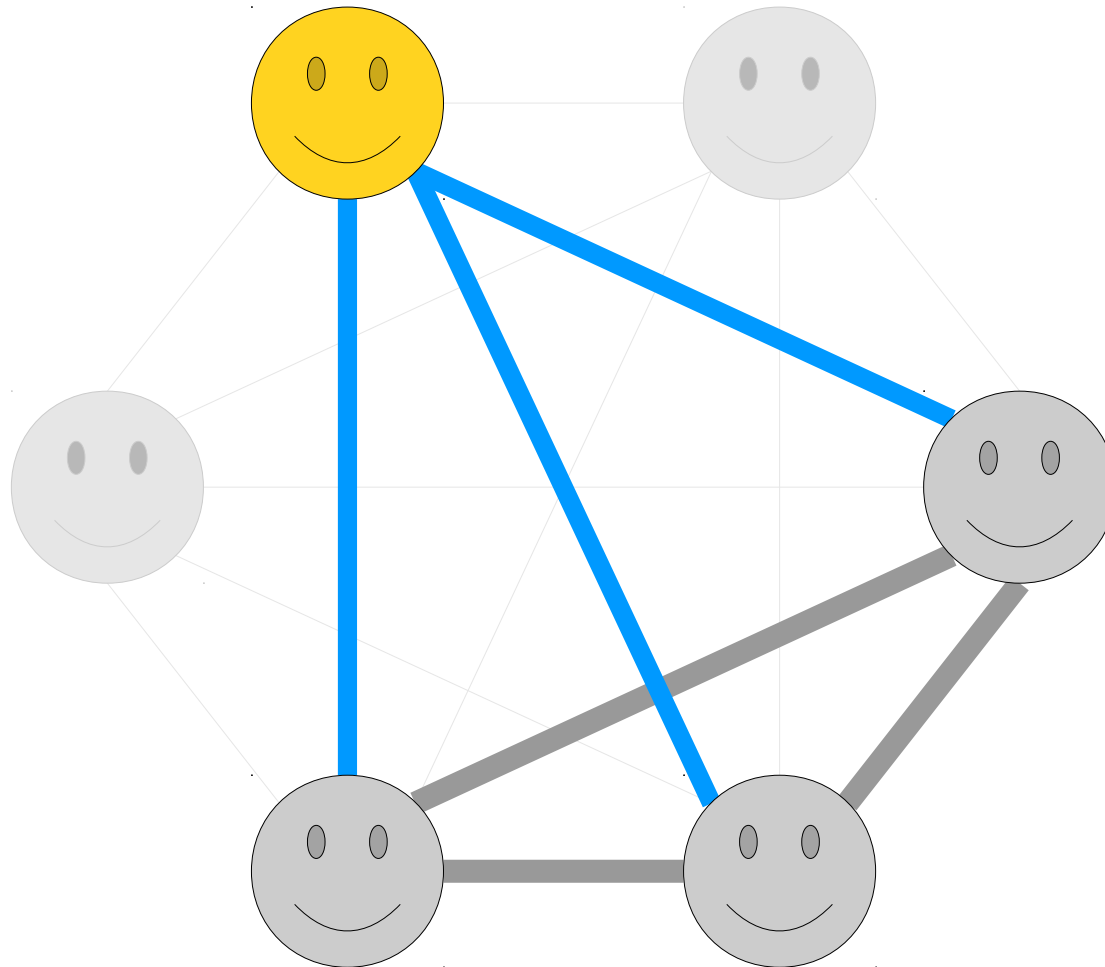


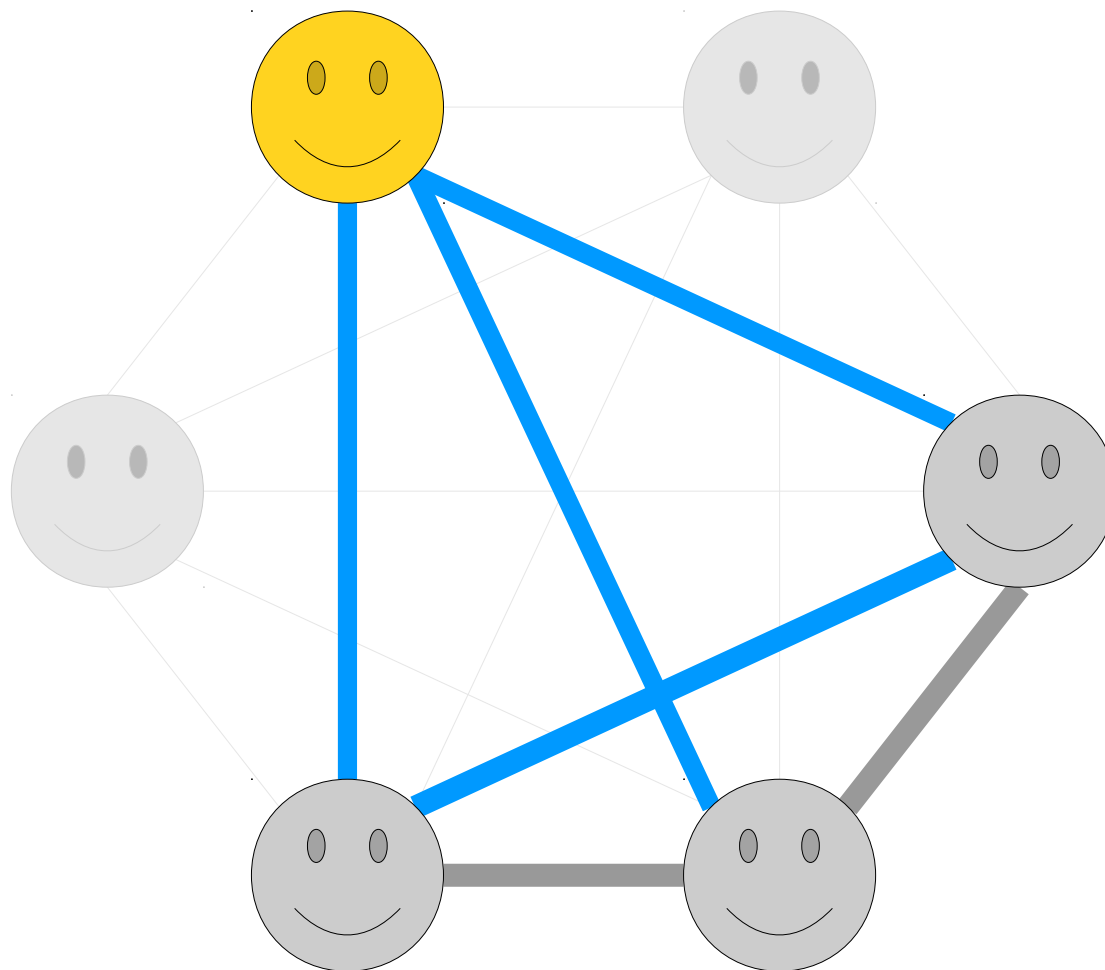


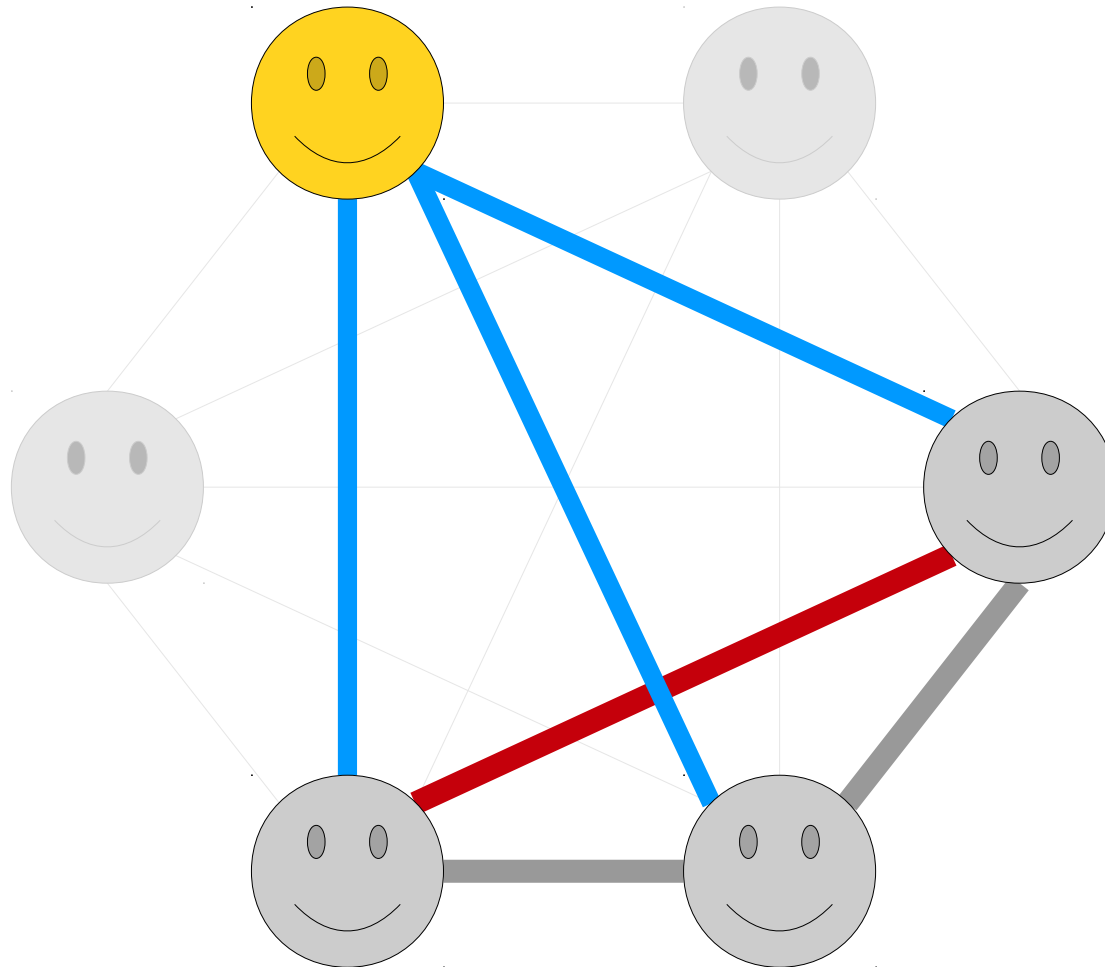


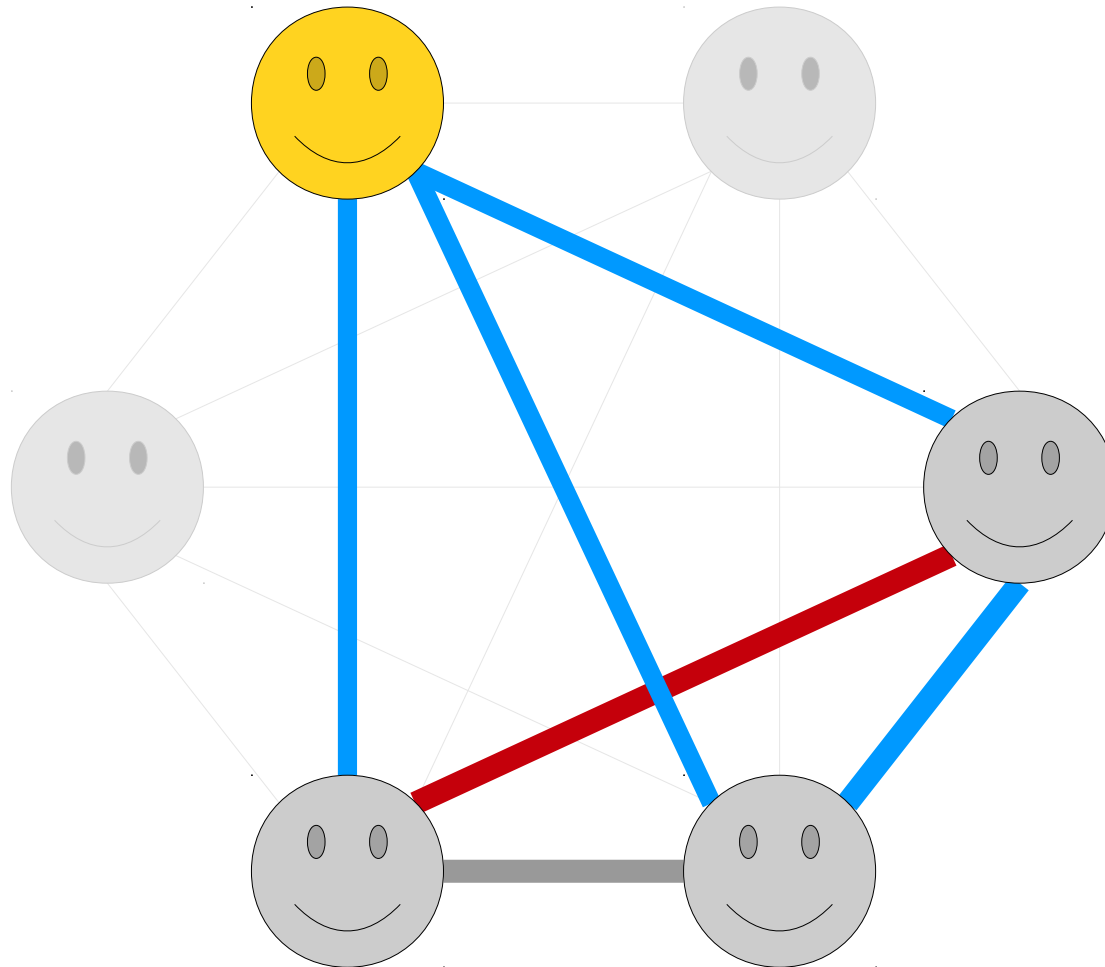
Observation 1: If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

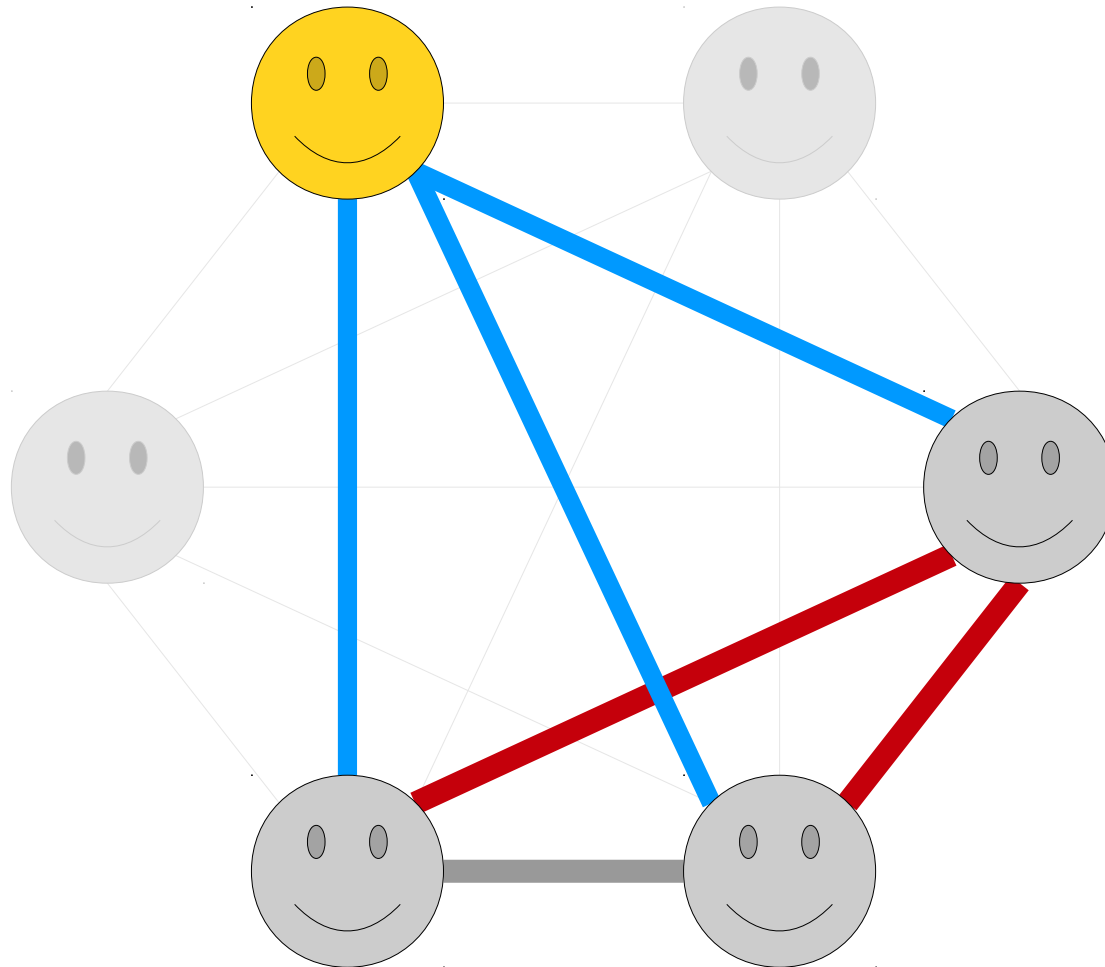


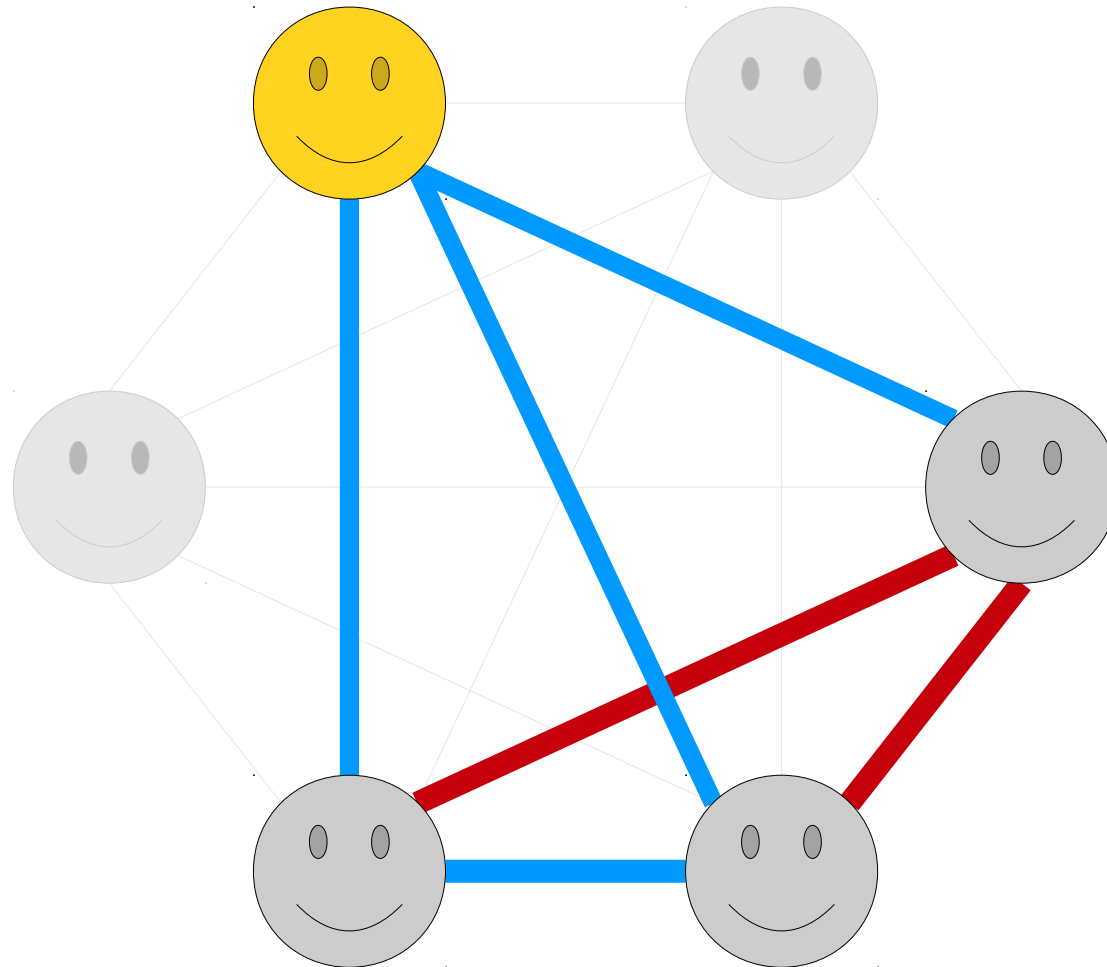


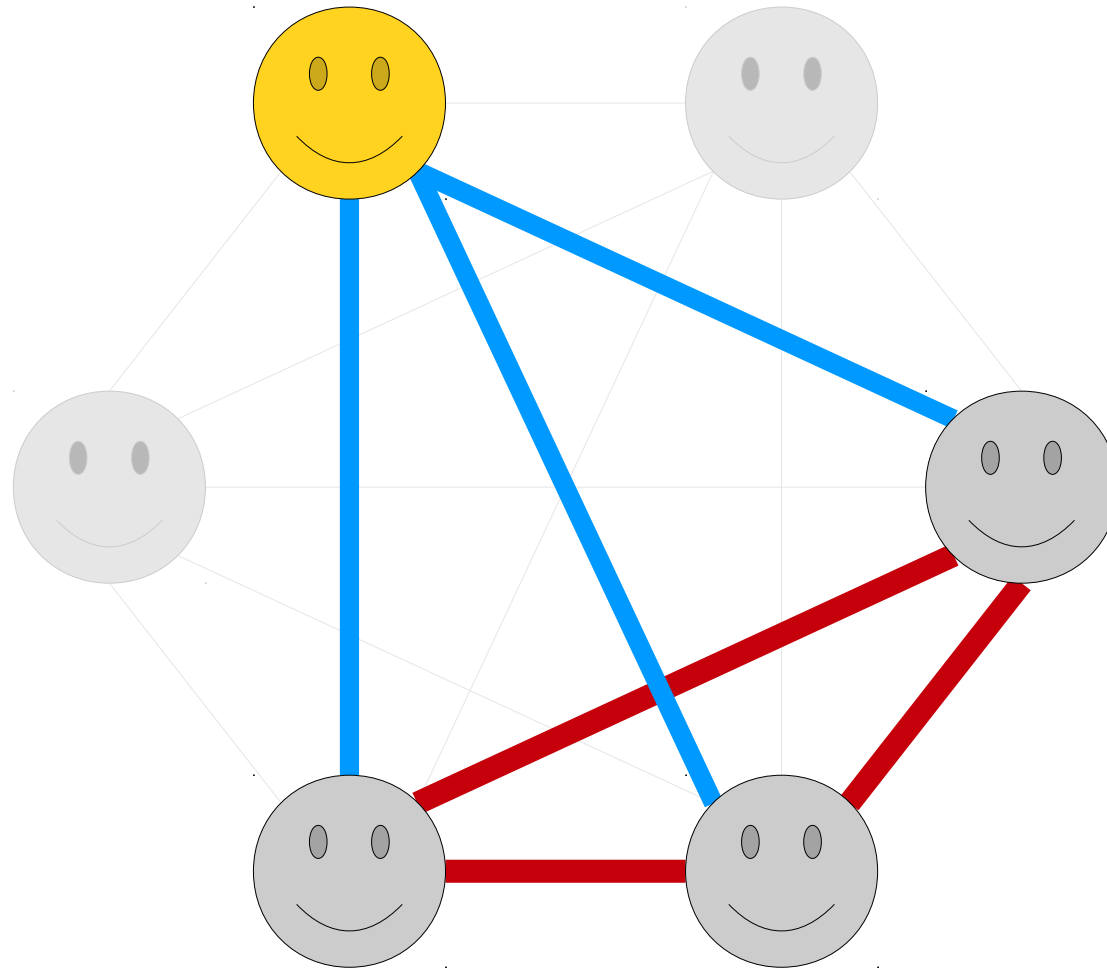












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Color the edges of the 6-clique either red or blue arbitrarily. Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2 .

Let r , s , and t be three of the nodes adjacent to node x along an edge of color c_1 . If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1 , then one of those edges plus the two edges connecting back to node x form a triangle of color c_1 . Otherwise, all three of those edges are of color c_2 , and they form a triangle of color c_2 . Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

- The proof we did is a special case of a broader result.
- ***Theorem (Ramsey's Theorem):*** For any natural number n , there is a smallest natural number $R(n)$ such that if the edges of an $R(n)$ -clique are colored red or blue, the resulting graph will contain either a red n -clique or a blue n -clique.
 - Our proof was that $R(3) \leq 6$.
- A more philosophical take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.