Mathematical Induction
Part Two
Problem Set Five

• Problem Set Four was due at 2:30PM today.

• Problem Set Five goes out today. It’s due next Friday at 2:30PM.
  • Play around with everything we’ve covered so far, plus a healthy dose of induction and inductive problem-solving.
Recap from Last Time
Let $P$ be some predicate. The principle of mathematical induction states that if

- $P(0)$ is true
- $\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}. P(n)$

...and it stays true...

...then it's always true.
**Theorem:** The sum of the first $n$ powers of two is $2^n - 1$.

**Proof:** Let $P(n)$ be the statement “the sum of the first $n$ powers of two is $2^n - 1$.” We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1.$$  \hspace{1cm} (1)

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k$$

$$= 2^k - 1 + 2^k \quad (via \ (1))$$

$$= 2(2^k) - 1$$

$$= 2^{k+1} - 1.$$

Therefore, $P(k + 1)$ is true, completing the induction. ■
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For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \quad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

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$$= 2^k - 1 + 2^k \quad (via \ (1))$$
$$= 2(2^k) - 1$$
$$= 2^{k+1} - 1.$$

Therefore, $P(k + 1)$ is true, completing the induction. ■
New Stuff!
Induction in Practice

• Often, a proof by induction will not explicitly state $P(n)$.
• Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
• Provided that there is sufficient detail to determine
  • what $P(n)$ is;
  • that $P(0)$ is true; and that
  • whenever $P(k)$ is true, $P(k+1)$ is true,
the proof is usually valid. In this class, you could err on the side of safety by always defining it, but it’s not required.
**Theorem:** The sum of the first \( n \) powers of two is \( 2^n - 1 \).

**Proof:** By induction.

For our base case, we'll prove the theorem is true when \( n = 0 \). The sum of the first zero powers of two is zero, and \( 2^0 - 1 = 0 \), so the theorem is true in this case.

For the inductive step, assume the theorem holds when \( n = k \) for some arbitrary \( k \in \mathbb{N} \). Then we have

\[
2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\
= 2^k - 1 + 2^k \\
= 2(2^k) - 1 \\
= 2^{k+1} - 1.
\]

So the theorem is true when \( n = k+1 \), completing the induction. ■
A Fun Application: The Limits of Data Compression
Bitstrings

- A **bitstring** is a finite sequence of 0s and 1s.
- Examples:
  - 11011100
  - 010101010101
  - 0000
  - $\varepsilon$ (the *empty string*)
- There are $2^n$ bitstrings of length $n$. 
Data Compression

- Inside a computer, all data are represented as sequences of 0s and 1s (bitstrings).
- To transfer data over a network (or on a flash drive, if you're still into that), it is useful to reduce the number of 0s and 1s before transferring it.
- Most real-world data can be compressed by exploiting redundancies.
  - Text repeats common patterns ("the", "and", etc.).
  - Bitmap images use similar colors throughout the image.
- **Idea:** Replace each bitstring with a shorter bitstring that contains all the original information.
  - This is called *lossless data compression.*
Compress

Transmit

1111010
Lossless Data Compression

- In order to losslessly compress data, we need two functions:
  - A *compression function* $C$, and
  - A *decompression function* $D$.
- We need to have $D(C(x)) = x$.
  - Otherwise, we can't uniquely encode or decode some bitstring.
- This means that $D$ must be a left inverse of $C$, so (as you proved in PS3!) $C$ must be injective.
A Perfect Compression Function

• Ideally, the compressed version of a bitstring would always be shorter than the original bitstring.

• **Question**: Can we find a lossless compression algorithm that always compresses a string into a shorter string?

• To handle the issue of the empty string (which can't get any shorter), let's assume we only care about strings of length at least 10.
A Counting Argument

- Let $\mathbb{B}^n$ be the set of bitstrings of length $n$, and $\mathbb{B}^{<n}$ be the set of bitstrings of length less than $n$.

- How many bitstrings of length $n$ are there?
  - *Answer*: $2^n$

- How many bitstrings of length *less than* $n$ are there?
  - *Answer*: $2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$

- By the pigeonhole principle, no function from $\mathbb{B}^n$ to $\mathbb{B}^{<n}$ can be injective – at least two elements must collide!

- Since a perfect compression function would have to be an injection from $\mathbb{B}^n$ to $\mathbb{B}^{<n}$, *there is no perfect compression function!*
Why this Result is Interesting

• Our result says that no matter how hard we try, it is impossible to compress every string into a shorter string.

• No matter how clever you are, you cannot write a lossless compression algorithm that always makes strings shorter.

• In practice, only highly redundant data can be compressed.

• The fields of information theory and Kolmogorov complexity explore the limits of compression; if you're interested, go explore!
Variations on Induction: *Starting Later*
Induction Starting at 0

• To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  • Show that $P(0)$ is true.
  • Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  • Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.
Induction Starting at $m$

• To prove that $P(n)$ is true for all natural numbers greater than or equal to $m$:
  • Show that $P(m)$ is true.
  • Show that for any $k \geq m$, that if $P(k)$ is true, then $P(k+1)$ is true.
  • Conclude $P(n)$ holds for all natural numbers greater than or equal to $m$. 
Variations on Induction: \textit{Bigger Steps}
Subdividing a Square
Subdividing a Square
Subdividing a Square
Subdividing a Square
Subdividing a Square
For what values of $n$ can a square be subdivided into $n$ squares?
Each of the original corners needs to be covered by a corner of the new smaller squares.
Each of the original corners needs to be covered by a corner of the new smaller squares.

Number of corners = 4

Number of squares < 4
Each of the original corners needs to be covered by a corner of the new smaller squares.

By the pigeonhole principle, at least one smaller square needs to cover at least two of the original square’s corners.
Number of corners = 4

Number of squares = 5
At least one square cannot be covering *any* of the original corners

Number of corners = 4

Number of squares = 5
The Key Insight
The Key Insight
The Key Insight
The Key Insight
The Key Insight
The Key Insight
The Key Insight
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- If we can subdivide a square into $n$ squares, we can also subdivide it into $n + 3$ squares.

- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into $n$ squares for any $n \geq 6$: 
  - For multiples of three, start with 6 and keep adding three squares until $n$ is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until $n$ is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until $n$ is reached.
**Theorem:** For any $n \geq 6$, it is possible to subdivide a square into $n$ smaller squares.
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**Proof:**
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**Proof:** Let \( P(n) \) be the statement “a square can be subdivided into \( n \) smaller squares.” We will prove by induction that \( P(n) \) holds for all \( n \geq 6 \), from which the theorem follows.
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As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares.
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As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
6 & 5 & 4 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & \\
6 & 7 & 3 \\
5 & 4 & \\
\end{array}
\quad
\begin{array}{cccc}
1 & & 2 \\
& 3 & \\
8 & 7 & 6 & 5 \\
\end{array}
\]
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For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that a square can be subdivided into $k$ squares.
**Theorem:** For any $n \geq 6$, it is possible to subdivide a square into $n$ smaller squares.

**Proof:** Let $P(n)$ be the statement “a square can be subdivided into $n$ smaller squares.” We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

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For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that a square can be subdivided into $k$ squares. We prove $P(k+3)$, that a square can be subdivided into $k+3$ squares.
**Theorem:** For any $n \geq 6$, it is possible to subdivide a square into $n$ smaller squares.

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Theorem: For any \( n \geq 6 \), it is possible to subdivide a square into \( n \) smaller squares.

Proof: Let \( P(n) \) be the statement “a square can be subdivided into \( n \) smaller squares.” We will prove by induction that \( P(n) \) holds for all \( n \geq 6 \), from which the theorem follows.

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**Theorem:** For any \( n \geq 6 \), it is possible to subdivide a square into \( n \) smaller squares.

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Why This Works

- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:
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\[ P(6) \rightarrow P(7) \rightarrow P(8) \]
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\[ P(k) \rightarrow P(k+3) \]

\[ P(8) \quad P(9) \quad P(10) \]
Why This Works

• This induction has three consecutive base cases and takes steps of size three.

• Thinking back to our “induction machine” analogy:

\[ P(8) \rightarrow P(k) \rightarrow P(k+3) \rightarrow P(9) \rightarrow P(10) \]
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\[ P(k) \rightarrow P(k+3) \]

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\[ P(11) \]
Why This Works

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\[ P(k) \rightarrow P(k+3) \]

\[ P(9) \]
\[ P(10) \]
\[ P(11) \]
Generalizing Induction

• When doing a proof by induction,
  • feel free to use multiple base cases, and
  • feel free to take steps of sizes other than one.

• Just be careful to make sure you cover all the numbers you think that you're covering!
  • We won't require that you prove you've covered everything, but it doesn't hurt to double-check!
More on Square Subdivisions

• There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.

• In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.

• Good starting resource: this Numberphile video on *Squaring the Square*. 
Complete Induction
Guess what!?
It’s time for Mathematical Calisthenics!
It’s time for Mathematical esthetics!
If you are the **leftmost** person in your row, stand up right now.

Everyone else: stand up as soon as the person to your left in your row stands up.

This is kinda like $P(0)$.

This is kinda like $P(k) \rightarrow P(k+1)$. 
Everyone, please be seated.
Let’s do this again... with a twist!
This is kinda like $P(0)$.

If you are the \textit{leftmost} person in your row, stand up right now.

Everyone else: stand up as soon as \textit{everyone} left of you in your row stands up.

What sort of sorcery is this?
Please be seated.

You all did a great job!
Let \( P \) be some predicate. The principle of complete induction states that if

\[
P(0) \text{ is true}
\]

and

for any \( k \in \mathbb{N} \), if \( P(0), P(1), \ldots, \text{and } P(k) \) are true, then \( P(k+1) \) is true

then

\[
\forall n \in \mathbb{N}. \ P(n)
\]

...and it stays true...

...then it's always true.
Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on $n$.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$. 
Complete Induction

• You can write proofs using the principle of **complete** induction as follows:
  
  • Define some predicate $P(n)$ to prove by induction on $n$.
  
  • Choose and prove a base case (probably, but not always, $P(0)$).
  
  • Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(0), P(1), P(2), \ldots, \text{and } P(k)$ are all true.
  
  • Prove $P(k+1)$.
  
  • Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.
A Motivating Example: *Rat Mazes*
Rat Mazes

- Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.
Rat Mazes

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• Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.

• The maze should have these properties:
  • There is one entrance and one exit in the border.
  • Every spot in the maze is reachable from every other spot.
  • There is exactly one path from each spot in the maze to each other spot.
Question: If you have an $n \times m$ grid of pegs, how many slats do you need to make?
A Special Type of Graph: *Trees*
A **tree** is a connected, nonempty graph with no simple cycles.

According to the above definition of trees, how many of these graphs are trees?
• A tree is a connected, nonempty graph with no simple cycles.
Trees

• A tree is a connected, nonempty graph with no simple cycles.

• Trees have tons of nice properties:
  • They're **maximally acyclic** (adding any missing edge creates a simple cycle)
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  - They're *maximally acyclic* (adding any missing edge creates a simple cycle)
  - They're *minimally connected* (deleting any edge disconnects the graph)

Proofs of these results are in the course reader if you're interested. They're also great exercises.
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- Trees have tons of nice properties:
  - They're *maximally acyclic* (adding any missing edge creates a simple cycle)
  - They're *minimally connected* (deleting any edge disconnects the graph)
- Proofs of these results are in the course reader if you're interested. They're also great exercises.
Trees

- **Theorem:** If $T$ is a tree with at least two nodes, then deleting any edge from $T$ splits $T$ into two nonempty trees $T_1$ and $T_2$.

- **Proof:** Left as an exercise to the reader. ☺
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Our Base Case
Assume any tree with at most $k$ nodes has one more node than edge.
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Consider an arbitrary tree with \( k+1 \) nodes.
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Consider an arbitrary tree with $k+1$ nodes.

Suppose there are $r$ nodes in the yellow tree.
Assume any tree with at most \( k \) nodes has one more node than edge.

Consider an arbitrary tree with \( k+1 \) nodes.

Suppose there are \( r \) nodes in the yellow tree.

Then there are \((k+1)-r\) nodes in the blue tree.
Assume any tree with at most $k$ nodes has one more node than edge.

Consider an arbitrary tree with $k+1$ nodes.

Suppose there are $r$ nodes in the yellow tree.

Then there are $(k+1)-r$ nodes in the blue tree.

There are $r-1$ edges in the yellow tree and $k-r$ edges in the blue tree.
Assume any tree with at most \( k \) nodes has one more node than edge.

Consider an arbitrary tree with \( k+1 \) nodes.

Suppose there are \( r \) nodes in the yellow tree.

Then there are \( (k+1) - r \) nodes in the blue tree.

There are \( r-1 \) edges in the yellow tree and \( k-r \) edges in the blue tree.
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Consider an arbitrary tree with $k+1$ nodes.

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Then there are $(k+1)-r$ nodes in the blue tree.

There are $r-1$ edges in the yellow tree and $k-r$ edges in the blue tree.

Adding in the initial edge we cut, there are $r-1 + k-r + 1 = k$ edges in the original tree.
**Theorem:** If $T$ is a tree with $n \geq 1$ nodes, then $T$ has $n-1$ edges.

**Proof:** Let $P(n)$ be the statement "any tree with $n$ nodes has $n-1$ edges." We will prove by induction that $P(n)$ holds for all $n \geq 1$, from which the theorem follows.

As a base case, we will prove $P(1)$, that any tree with 1 node has 0 edges. Any such tree has a single node, so it cannot have any edges.

Now, assume for some arbitrary $k \geq 1$ that $P(1), P(2), \ldots, P(k)$ are true, so any tree with between 1 and $k$ nodes has one more node than edge. We will prove $P(k+1)$, that any tree with $k+1$ nodes has $k$ edges.

Consider any tree $T$ with $k+1$ nodes. Since $T$ has at least two nodes and is connected, it must contain at least one edge. Choose any edge in $T$ and delete it. This splits $T$ into two nonempty trees $T_1$ and $T_2$.

Every edge in $T$ is part of $T_1$, is part of $T_2$, or is the initial edge we deleted. Let $r$ be the number of nodes in $T_1$. Since every node in $T$ belongs to either $T_1$ or $T_2$, we see that $T_2$ has $(k+1)-r$ nodes. Additionally, since $T_1$ and $T_2$ are nonempty, neither $T_1$ nor $T_2$ contains all the nodes from $T$.

Therefore, $T_1$ and $T_2$ each have between 1 and $k$ nodes. We can then apply our inductive hypothesis to see that $T_1$ has $r-1$ edges and $T_2$ has $k-r$ edges. Thus the total number of edges in $T$ is $1 + (r-1) + (k-r) = k$, as required. Therefore, $P(k+1)$ is true, completing the induction. $\blacksquare$
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Induction vs. Complete Induction

Smaller problem → Bigger problem
Induction vs. Complete Induction

Sum of first $k$ powers of 2 $= 2^k - 1$

Induction

Sum of first $k+1$ powers of 2 $= 2^{k+1} - 1$
Induction vs. Complete Induction

**Induction**

Sum of first $k$ powers of 2, \( 2 = 2^k - 1 \)

Complete Induction

Sum of first $k+1$ powers of 2, \( 2 = 2^{k+1} - 1 \)
Induction vs. Complete Induction

Induction

Size = $k$

Complete Induction

Size = ??

Size = $k+1$

Size = $k+1$
Rat Mazes

• Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.

• **Question:** How many slats do you need to create?

\[mn - 2\]
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This is a tree!
• Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.

• **Question:** How many slats do you need to create?

• **Answer:** $mn - 2$. 

This is a tree!
For more on trees, take CS161 / 261 / 267!
An Important Milestone
Recap: *Discrete Mathematics*

- The past five weeks have focused exclusively on discrete mathematics:

  - Induction
  - Functions
  - Graphs
  - The Pigeonhole Principle
  - Relations
  - Mathematical Logic
  - Set Theory
  - Cardinality

- These are building blocks we will use throughout the rest of the quarter.

- These are building blocks you will use throughout the rest of your CS career.
Next Up: *Computability Theory*

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- Get ready to explore the boundaries of what computers could ever be made to do.
Next Time

- **Formal Language Theory**
  - How are we going to formally model computation?

- **Finite Automata**
  - A simple but powerful computing device made entirely of math!

- **DFAs**
  - A fundamental building block in computing.