Complexity Theory
Part One
It may be that since one is customarily concerned with existence, [...] finiteness, and so forth, one is not inclined to take seriously the question of the existence of a better-than-finite algorithm.

- Jack Edmonds, “Paths, Trees, and Flowers”
WELCOME TO THEORYLAND
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- Jack Edmonds, “Paths, Trees, and Flowers”
A Decidable Problem

- **Presburger arithmetic** is a logical system for reasoning about arithmetic.
  - $\forall x. \ x + 1 \neq 0$
  - $\forall x. \forall y. (x + 1 = y + 1 \rightarrow x = y)$
  - $\forall x. \ x + 0 = x$
  - $\forall x. \forall y. (x + y) + 1 = x + (y + 1)$
  - $(P(0) \land \forall y. (P(y) \rightarrow P(y + 1))) \rightarrow \forall x. \ P(x)$

- Given a statement, it is decidable whether that statement can be proven from the laws of Presburger arithmetic.

- Any Turing machine that decides whether a statement in Presburger arithmetic is true or false has to move its tape head at least $2^{2^{cn}}$ times on some inputs of length $n$ (for some fixed constant $c \geq 1$).
For Reference

- Assume $c = 1$. 
The Limits of Decidability

• The fact that a problem is decidable does not mean that it is feasibly decidable.

• In *computability theory*, we ask the question
  What problems can be solved by a computer?

• In *complexity theory*, we ask the question
  What problems can be solved *efficiently* by a computer?

• In the remainder of this course, we will explore this question in more detail.
Where We've Been

- The class $\mathbf{R}$ represents problems that can be solved by a computer.
- The class $\mathbf{RE}$ represents problems where “yes” answers can be verified by a computer.
Where We're Going

• The class $\mathbf{P}$ represents problems that can be solved *efficiently* by a computer.

• The class $\mathbf{NP}$ represents problems where “yes” answers can be verified *efficiently* by a computer.
Regular Languages

CFLs

R

RE

All Languages
Undecidable Languages

Regular Languages

Efficiently Decidable Languages

CFLs

Undecidable Languages
The Setup

- In order to study computability, we needed to answer these questions:
  - What is “computation?”
  - What is a “problem?”
  - What does it mean to “solve” a problem?
- To study complexity, we need to answer these questions:
  - What does “complexity” even mean?
  - What is an “efficient” solution to a problem?
Measuring Complexity

• Suppose that we have a decider \( D \) for some language \( L \).
• How might we measure the complexity of \( D \)?
Measuring Complexity

- Suppose that we have a decider $D$ for some language $L$.
- How might we measure the complexity of $D$?
  - Number of states.
  - Size of tape alphabet.
  - Size of input alphabet.
  - Amount of tape required.
  - Amount of time required.
  - Number of times a given state is entered.
  - Number of times a given symbol is printed.
  - Number of times a given transition is taken.
  - (Plus a whole lot more...)
Measuring Complexity

- Suppose that we have a decider $D$ for some language $L$.
- How might we measure the complexity of $D$?
  - Number of states.
  - Size of tape alphabet.
  - Size of input alphabet.
  - Amount of tape required.
  - **Amount of time required.**
    - Number of times a given state is entered.
    - Number of times a given symbol is printed.
    - Number of times a given transition is taken.
    (Plus a whole lot more...)
What is an efficient algorithm?
Searching Finite Spaces

• Many decidable problems can be solved by searching over a large but finite space of possible options.

• Searching this space might take a staggeringly long time, but only finite time.

• From a decidability perspective, this is totally fine.

• From a complexity perspective, this may be totally unacceptable.
A Sample Problem

| 4 | 3 | 11 | 9 | 7 | 13 | 5 | 6 | 1 | 12 | 2 | 8 | 0 | 10 |
A Sample Problem

Goal: Find the length of the longest increasing subsequence of this sequence.

4  3  11  9  7  13  5  6  1  12  2  8  0  10

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A Sample Problem

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4 3 11 9 7 13 5 6 1 12 2 8 0 10

Goal: Find the length of the longest increasing subsequence of this sequence.
Longest Increasing Subsequences

- **One possible algorithm:** try all subsequences, find the longest one that's increasing, and return that.
- There are $2^n$ subsequences of an array of length $n$.
  - (Each subset of the elements gives back a subsequence.)
- Checking all of them to find the longest increasing subsequence will take time $O(n \cdot 2^n)$.
- Nifty fact: the age of the universe is about $4.3 \times 10^{26}$ nanoseconds old. That's about $2^{85}$ nanoseconds.
- Practically speaking, this algorithm doesn't terminate if you give it an input of size 100 or more.
A Different Approach
Patience Sorting

Place each number on top of a pile. Put each number on top of the first pile whose top value is larger than it. (If you can’t, make a new pile.) Then, add a link to the top number in the previous pile.
Trace backwards from the top of the last pile. The numbers you visit form one of the longest increasing subsequences of your original sequence.
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Trace backwards from the top of the last pile. The numbers you visit form one of the longest increasing subsequences of your original sequence.
Longest Increasing Subsequences

• **Theorem:** There is an algorithm that can find the longest increasing subsequence of an array in time $O(n^2)$.
  • It’s the previous *patience sorting* algorithm, with some clever implementation tricks.

• This algorithm works by exploiting particular aspects of how longest increasing subsequences are constructed. It's not immediately obvious that it works correctly.

• **Phenomenal Exercise 1:** Prove that this procedure always works!

• **Phenomenal Exercise 2:** Show that you can actually implement this same algorithm in time $O(n \log n)$. 
Another Problem

A

B

C

D

E

F
Another Problem
Another Problem

Goal: Determine the length of the shortest path from A to F in this graph.
Shortest Paths

- It is possible to find the shortest path in a graph by listing off all sequences of nodes in the graph in ascending order of length and finding the first that's a path.
- This takes time $O(n \cdot n!)$ in an $n$-node graph.
- For reference: 29! nanoseconds is longer than the lifetime of the universe.
Shortest Paths

- **Theorem:** It's possible to find the shortest path between two nodes in an $n$-node, $m$-edge graph in time $O(m + n)$.

- **Proof idea:** Use breadth-first search!

- The algorithm is a bit nuanced. It uses some specific properties of shortest paths and the proof of correctness is nontrivial.
For Comparison

- **Longest increasing subsequence:**
  - Naive: $O(n \cdot 2^n)$
  - Fast: $O(n^2)$

- **Shortest path problem:**
  - Naive: $O(n \cdot n!)$
  - Fast: $O(n + m)$. 
Defining Efficiency

- When dealing with problems that search for the “best” object of some sort, there are often at least exponentially many possible options.
- Brute-force solutions tend to take at least exponential time to complete.
- Clever algorithms often run in time $O(n)$, or $O(n^2)$, or $O(n^3)$, etc.
Polynomials and Exponentials

• An algorithm runs in *polynomial time* if its runtime is some polynomial in $n$.
  • That is, time $O(n^k)$ for some constant $k$.
• Polynomial functions “scale well.”
  • Small changes to the size of the input do not typically induce enormous changes to the overall runtime.
• Exponential functions scale terribly.
  • Small changes to the size of the input induce huge changes in the overall runtime.
A language $L$ can be \textit{decided efficiently} if there is a TM that decides it in polynomial time.

Equivalently, $L$ can be decided efficiently if it can be decided in time $O(n^k)$ for some $k \in \mathbb{N}$.

Like the Church-Turing thesis, this is \textit{not} a theorem!

It's an assumption about the nature of efficient computation, and it is somewhat controversial.
The Cobham-Edmonds Thesis

- Efficient runtimes:
  - $4n + 13$
  - $n^3 - 2n^2 + 4n$
  - $n \log \log n$
- “Efficient” runtimes:
  - $n^{1,000,000,000,000,000}$
  - $10^{500}$

- Inefficient runtimes:
  - $2^n$
  - $n!$
  - $n^n$
- “Inefficient” runtimes:
  - $n^{0.0001 \log n}$
  - $1.000000001^n$
Why Polynomials?

- Polynomial time *somewhat* captures efficient computation, but has a few edge cases.
- However, polynomials have very nice mathematical properties:
  - The sum of two polynomials is a polynomial. (Running one efficient algorithm, then another, gives an efficient algorithm.)
  - The product of two polynomials is a polynomial. (Running one efficient algorithm a “reasonable” number of times gives an efficient algorithm.)
  - The *composition* of two polynomials is a polynomial. (Using the output of one efficient algorithm as the input to another efficient algorithm gives an efficient algorithm.)
The Complexity Class \( \mathbf{P} \)

- The *complexity class \( \mathbf{P} \) (for \( p \)olynomial time)* contains all problems that can be solved in polynomial time.
- Formally:
  \[
  \mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}
  \]
- Assuming the Cobham-Edmonds thesis, a language is in \( \mathbf{P} \) if it can be decided efficiently.
Examples of Problems in $\mathbf{P}$

- All regular languages are in $\mathbf{P}$.
  - All have linear-time TMs.
- All CFLs are in $\mathbf{P}$.
  - Requires a more nuanced argument (the CYK algorithm or Earley's algorithm.)
- And a ton of other problems are in $\mathbf{P}$ as well.
  - Curious? Take CS161!
Undecidable Languages

Regular Languages

CFLs

Efficiently Decidable Languages

Undecidable Languages
Undecidable Languages

Regular Languages

CFLs

P

R

Undecidable Languages
What *can't* you do in polynomial time?
How many simple paths are there from the start node to the end node?
How many subsets of this set are there?
An Interesting Observation

• There are (at least) exponentially many objects of each of the preceding types.

• However, each of those objects is not very large.
  • Each simple path has length no longer than the number of nodes in the graph.
  • Each subset of a set has no more elements than the original set.

• This brings us to our next topic...
What if you need to search a large space for a single object?
Does this Sudoku problem have a solution?
Does this Sudoku problem have a solution?
Verifiers – Again

Is there an ascending subsequence of length at least 7?
Verifiers – Again

Is there an ascending subsequence of length at least 7?
Verifiers – Again

Is there a simple path that goes through every node exactly once?
Verifiers – Again

Is there a simple path that goes through every node exactly once?
Verifiers

- Recall that a **verifier** for $L$ is a TM $V$ such that
  - $V$ halts on all inputs.
  - $w \in L$ iff $\exists c \in \Sigma^*. V$ accepts $\langle w, c \rangle$. 
Polynomial-Time Verifiers

- A **polynomial-time verifier** for $L$ is a TM $V$ such that
  - $V$ halts on all inputs.
  - $w \in L$ iff $\exists c \in \Sigma^*$. $V$ accepts $\langle w, c \rangle$.
  - $V$'s runtime is a polynomial in $|w|$ (that is, $V$'s runtime is $O(|w|^k)$ for some integer $k$)
The Complexity Class **NP**

- The complexity class **NP** (*nondeterministic polynomial time*) contains all problems that can be verified in polynomial time.
- Formally:
  \[
  \textbf{NP} = \{ \, L \mid \text{There is a polynomial-time verifier for } L \, \}
  \]
- The name **NP** comes from another way of characterizing **NP**. If you introduce *nondeterministic Turing machines* and appropriately define “polynomial time,” then **NP** is the set of problems that an NTM can solve in polynomial time.
- **Useful fact:** **NP** $\subseteq \textbf{R}$. Come talk to me after class if you’re curious why!
\[ P = \{ L \mid \text{there is a polynomial-time decider for } L \} \]

\[ NP = \{ L \mid \text{there is a polynomial-time verifier for } L \} \]
\[
\text{R} = \{ L \mid \text{there is a polynomial-time decider for } L \} \\
\text{RE} = \{ L \mid \text{there is a polynomial-time verifier for } L \}
\]
We know that $\mathbb{R} \neq \mathbb{RE}$.

So does that mean $\mathbb{P} \neq \mathbb{NP}$?
Time-Out for Announcements!
Problem Sets

• Problem Set Six was due today at 3:00PM.
• Problem Set Seven is due next Wednesday at 3:00PM.
  • As a reminder, *no late submissions will be accepted*. Please budget enough time to get your submission in!
  • *Very smart idea*: submit at least two hours early.
• As always, feel free to ask questions in office hours or online via Piazza.
  • Note: Updated OH schedule for next Tuesday and Wednesday.
Final Exam Logistics

• Our final exam is Friday, August 16\textsuperscript{th} from 7PM – 10PM in \textit{Bishop Auditorium}.

• The exam is cumulative. You’re responsible for topics from PS0 – PS7 and all of the lectures up through and including Unsolvable Problems.

• The exam is closed-book, closed-computer, and limited-note. You can bring one double-sided sheet of 8.5” × 11” notes with you to the exam, decorated any way you’d like.

• Students with OAE accommodations: if we don’t yet have your OAE letter, please send it to us ASAP.
Preparing for the Exam

• We’ve posted two practice final exams, with solutions, to the course website. They’re on the *Extra Practice* page under *Resources*.
  • The practice exam we’ll be using during the practice final will be released on Wednesday.
• *Review Session* on Monday, August 12\(^{th}\) here during class, led by your lovely TAs!
• *Practice Final* on Wednesday, August 14\(^{th}\) from 5:30-8:30 PM upstairs in Gates 104.
Back to CS103!
And now...
The 

Biggest Question 

in 

Theoretical Computer Science
\[ P = ? = NP \]
\[ \mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \} \]

\[ \mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \} \]

**Diagram:**
- **Input string** \((w)\)
- **Polynomial-Time Decider for** \(L\)
- **Output:**
  - **Yes!**
  - **No!**
\[ \mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \} \]

\[ \mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \} \]

\[ \mathbf{P} \subseteq \mathbf{NP} \]
Which Picture is Correct?
Which Picture is Correct?

P  NP
\[
\text{P} \equiv \text{NP}
\]

- The \( \text{P} \not= \text{NP} \) question is the most important question in theoretical computer science.
- With the verifier definition of \( \text{NP} \), one way of phrasing this question is
  
  \textit{If a solution to a problem can be checked efficiently, can that problem be solved efficiently?}

- An answer either way will give fundamental insights into the nature of computation.
Why This Matters

- The following problems are known to be efficiently verifiable, but have no known efficient solutions:
  - Determining whether an electrical grid can be built to link up some number of houses for some price (Steiner tree problem).
  - Determining whether a simple DNA strand exists that multiple gene sequences could be a part of (shortest common supersequence).
  - Determining the best way to assign hardware resources in a compiler (optimal register allocation).
  - Determining the best way to distribute tasks to multiple workers to minimize completion time (job scheduling).
  - *And many more.*
- If $P = NP$, *all* of these problems have efficient solutions.
- If $P \neq NP$, *none* of these problems have efficient solutions.
Why This Matters

• If $P = NP$:
  • A huge number of seemingly difficult problems could be solved efficiently.
  • Our capacity to solve many problems will scale well with the size of the problems we want to solve.

• If $P \neq NP$:
  • Enormous computational power would be required to solve many seemingly easy tasks.
  • Our capacity to solve problems will fail to keep up with our curiosity.
What We Know

• Resolving $\mathbf{P} \neq \mathbf{NP}$ has proven extremely difficult.

• In the past 45 years:
  • Not a single correct proof either way has been found.
  • Many types of proofs have been shown to be insufficiently powerful to determine whether $\mathbf{P} \neq \mathbf{NP}$.
  • A majority of computer scientists believe $\mathbf{P} \neq \mathbf{NP}$, but this isn't a large majority.

• Interesting read: Interviews with leading thinkers about $\mathbf{P} \neq \mathbf{NP}$:
  • http://web.ing.puc.cl/~jabdator/ic2212/poll-1.pdf
The Million-Dollar Question

The Clay Mathematics Institute has offered a $1,000,000 prize to anyone who proves or disproves $P = NP$. 
“My hunch is that $[\text{P} \not= \text{NP}]$ will be solved by a young researcher who is not encumbered by too much conventional wisdom about how to attack the problem.”

– Prof. Richard Karp

(The guy who first popularized the P ≠ NP problem.)
“There is something very strange about this problem, something very philosophical. It is the greatest unsolved problem in mathematics [...] It is the *raison d’être* of abstract computer science, and as long as it remains unsolved, its mystery will ennoble the field.”

-Prof. Jim Owings

*(Computability/Complexity theorist)*
What do we know about $\mathbf{P} = \mathbf{NP}$?
Adapting our Techniques
\( P = \{ L \mid \text{there is a polynomial-time decider for } L \} \)

\( \text{NP} = \{ L \mid \text{there is a polynomial-time verifier for } L \} \)
R = \{ L | \text{there is a polynomial-time decider for } L \ \}\}

RE = \{ L | \text{there is a polynomial-time verifier for } L \ \}\}
We know that $\mathbb{R} \neq \mathbb{RE}$.

So does that mean $\mathbb{P} \neq \mathbb{NP}$?
A Problem

• The \textbf{R} and \textbf{RE} languages correspond to problems that can be decided and verified, \textit{period}, without any time bounds.

• To reason about what's in \textbf{R} and what's in \textbf{RE}, we used two key techniques:
  • \textit{Universality}: TMs can run other TMs as subroutines.
  • \textit{Self-Reference}: TMs can get their own source code.

• Why can't we just do that for \textbf{P} and \textbf{NP}?
Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve \( P \neq NP \).

Proof: Take CS154!
So how *are* we going to reason about P and NP?
Problems in \textbf{NP} vary widely in their difficulty, even if \textbf{P} = \textbf{NP}.

How can we rank the relative difficulties of problems?
Reducibility
Maximum Matching

- Given an undirected graph $G$, a *matching* in $G$ is a set of edges such that no two edges share an endpoint.

- A *maximum matching* is a matching with the largest number of edges.
Maximum Matching

- Given an undirected graph $G$, a \textit{matching} in $G$ is a set of edges such that no two edges share an endpoint.

- A \textbf{maximum matching} is a matching with the largest number of edges.
Maximum Matching

• Given an undirected graph $G$, a matching in $G$ is a set of edges such that no two edges share an endpoint.

• A maximum matching is a matching with the largest number of edges.

A matching, but not a maximum matching.
Maximum Matching

- Given an undirected graph $G$, a **matching** in $G$ is a set of edges such that no two edges share an endpoint.

- A **maximum matching** is a matching with the largest number of edges.

A maximum matching.
Maximum Matching

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Maximum Matching

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Maximum Matching

• Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  • He’s the guy with the quote about “better than decidable.”

• Using this fact, what other problems can we solve?
Domino Tiling
Domino Tiling
Domino Tiling
Domino Tiling
Domino Tiling
Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
In Pseudocode

```java
boolean canPlaceDominoes(Grid G, int k) {
    return hasMatching(gridToGraph(G), k);
}
```
**Intuition:**

Tiling a grid with dominoes can't be "harder" than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.
Another Example
Reachability

• Consider the following problem:

  Given an directed graph $G$ and nodes $s$ and $t$ in $G$, is there a path from $s$ to $t$?

• This problem can be solved in polynomial time (use DFS or BFS).
Converter Conundrums

• Suppose that you want to plug your laptop into a projector.
• Your laptop only has a VGA output, but the projector needs HDMI input.
• You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
• **Question:** Can you plug your laptop into the projector?
Converter Conundrums

**Connectors**
- RGB to USB
- VGA to DisplayPort
- DB13W3 to CATV
- DisplayPort to RGB
- DB13W3 to HDMI
- DVI to DB13W3
- S-Video to DVI
- FireWire to SDI
- VGA to RGB
- DVI to DisplayPort
- USB to S-Video
- SDI to HDMI
Converter Conundrums

Connectors
- RGB to USB
- VGA to DisplayPort
- DB13W3 to CATV
- DisplayPort to RGB
- DB13W3 to HDMI
- DVI to DB13W3
- S-Video to DVI
- FireWire to SDI
- VGA to RGB
- DVI to DisplayPort
- USB to S-Video
- SDI to HDMI
Converter Conundrums

- VGA
- RGB
- USB
- DisplayPort
- DB13W3
- CATV
- HDMI
- DVI
- S-Video
- FireWire
- SDI
Converter Conundrums

- VGA
- RGB
- USB
- DisplayPort
- DB13W3
- CATV
- HDMI
- DVI
- S-Video
- FireWire
- SDI
Converter Conundrums

VGA → RGB → USB

DisplayPort → VGA

DB13W3 → CATV

HDMI → HDMI

DVI → S-Video

FireWire → SDI
Converter Conundrums

Connectors
RGB to USB
VGA to DisplayPort
DB13W3 to CATV
DisplayPort to RGB
DB13W3 to HDMI
DVI to DB13W3
S-Video to DVI
FireWire to SDI
VGA to RGB
DVI to DisplayPort
USB to S-Video
SDI to HDMI
In Pseudocode

```java
boolean canPlugIn(List<Plug> plugs) {
    return isReachable(plugsToGraph(plugs), VGA, HDMI);
}
```
**Intuition:**

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.
Intuition:

Problem A can't be "harder" than problem B, because solving problem B lets us solve problem A.

```cpp
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```
bool solveProblemA(string input) {
    return solveProblemB(transform(transform(input)));
}

- If $A$ and $B$ are problems where it's possible to solve problem $A$ using the strategy shown above*, we write

$$A \leq_p B.$$ 

- We say that $A$ is polynomial-time reducible to $B$.

* Assuming that $\text{transform}$ runs in polynomial time.
bool solveProblemA(string input) {
    return solveProblemB(transform(transform(input)));
}

• This is a powerful general problem-solving technique. You’ll see it a lot in CS161.
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$. 
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbb{P}$, then $A \in \mathbb{P}$. 
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$. 
Polynomial-Time Reductions

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Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.
This $\leq_p$ relation lets us rank the relative difficulties of problems in $\mathbf{P}$ and $\mathbf{NP}$.

What else can we do with it?
Next Time

- **NP-Completeness**
  - What are the hardest problems in NP?