Unsolvable Problems

Part Two
Outline for Today

• **More on Undecidability**
  • Even more problems we can’t solve.

• **A Different Perspective on RE**
  • What exactly does “recognizability” mean?

• **Verifiers**
  • A new approach to problem-solving.

• **Beyond RE**
  • A beautiful example of an impossible problem.
Recap from Last Time
```c++
bool willAccept(string function, string input) {
    // Returns true if function(input) returns true.
    // Returns false otherwise.
}

bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```
**Theorem:** \( A_{TM} \notin R. \)

**Proof:** By contradiction; assume that \( A_{TM} \in R \). Then there is a decider \( D \) for \( A_{TM} \). We can represent \( D \) as a function

\[
\text{bool willAccept(string function, string w)};
\]

that takes in the source code of a function \( function \) and a string \( w \), then returns true if \( function(w) \) returns true and returns false otherwise. Given this, consider this function \( \text{trickster} \):

\[
\text{bool trickster(string input)} \{
    \text{string me = /* source code of trickster */};
    \text{return !willAccept(me, input);} 
\}
\]

Since \( \text{willAccept} \) decides \( A_{TM} \) and \( me \) holds the source of \( \text{trickster} \), we know that \( \text{willAccept(me, input)} \) returns true if and only if \( \text{trickster(input)} \) returns true. Given how \( \text{trickster} \) is written, we see that

\( \text{willAccept(me, input)} \) returns true if and only if \( \text{trickster(input)} \) returns false.

This means that

\( \text{trickster(input)} \) returns true if and only if \( \text{trickster(input)} \) returns false.

This is impossible. We’ve reached a contradiction, so our assumption was wrong and \( A_{TM} \) is undecidable. \( \blacksquare \)
New Stuff!
More Impossibility Results
The Halting Problem

• The most famous undecidable problem is the **halting problem**, which asks:

  Given a TM $M$ and a string $w$, will $M$ halt when run on $w$?

• As a formal language, this problem would be expressed as

  $HALT = \{ \langle M, w \rangle | M$ is a TM that halts on $w \}$

• **Theorem:** $HALT$ is recognizable, but undecidable.
  • There’s a recognizer for $HALT$.
  • There is no decider for $HALT$. 
**Claim:** \( \text{HALT} \in \text{RE} \).

**Idea:** If you were certain that a TM \( M \) halted on a string \( w \), could you convince me of that?

Yes – just run \( M \) on \( w \) and see what happens!

```cpp
bool willHalt(string TM, string w) {
    set up a simulation of \( M \) running on \( w \);
    while (true) {
        if (\( M \) returned true) return true;
        else if (\( M \) returned false) return true;
        else simulate one more step of \( M \) running on \( w \);
    }
}
```
**Theorem:** The halting problem is undecidable.
A Decider for $\text{HALT}$

- Let’s suppose that, somehow, we managed to build a decider for $\text{HALT} = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on } w \}$. 
- Schematically, that decider would look like this:

  ![Decider Diagram]

  - $M$ and $w$ go into the decider box.
  - The decider box outputs "Yes, $M$ halts on $w$." or "No, $M$ loops on $w$." depending on whether $M$ halts or loops on $w$.

- We could represent this decider in software as a method

  ```cpp
  bool willHalt(string function, string input);
  ```

  that takes as input a function `function` and a string `input`, then
  - returns true if `function(input)` returns anything (halts), and
  - returns false if `function(input)` never returns anything (loops).
bool willHalt(string function, string input) {
    // Returns true if function(input) halts.
    // Returns false otherwise.
}

bool trickster(string input) {

    trickster(input) halts
    ↔
    willHalt(me, input) returns true
bool willHalt(string function, string input) {
    // Returns true if function(input) halts.
    // Returns false otherwise.
}

bool trickster(string input) {
    string me = /* source code of trickster */;
    if (willHalt(me, input)) {
        while (true) {
            // Do nothing
        }
    } else {
        return true;
    }
}
**Theorem:** $\text{HALT} \notin R$.

**Proof:** By contradiction; assume that $\text{HALT} \in R$. Then there is a decider $D$ for $\text{HALT}$. We can represent $D$ as a function

```cpp
bool willHalt(string function, string w);
```

that takes in the source code of a function $\text{function}$ and a string $w$, then returns true if $\text{function}(w)$ halts and returns false otherwise. Given this, consider this function $\text{trickster}$:

```cpp
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (willHalt(me, input)) {
        while (true) {
        }
    } else {
        return true;
    }
    return true;
}
```

Since $\text{willHalt}$ decides $\text{HALT}$ and $me$ holds the source of $\text{trickster}$, we know that

$\text{willHalt}(me, input)$ returns true if and only if $\text{trickster}(input)$ halts.

Given how $\text{trickster}$ is written, we see that

$\text{willHalt}(me, input)$ returns true if and only if $\text{trickster}(input)$ loops.

This means that

$\text{trickster}(input)$ halts if and only if $\text{trickster}(input)$ loops.

This is impossible. We’ve reached a contradiction, so our assumption was wrong and $\text{HALT}$ is undecidable. ■
So What?

- These problems might not seem all that exciting, so who cares if we can't solve them?
- Turns out, this same line of reasoning can be used to show that some very important problems are impossible to solve.
Analogy Time!
Engineering Problem: Design a diesel engine that doesn’t emit lots of NO\textsubscript{x} pollutants.
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*Engineering Prowess!*
**Engineering Problem:** Design a diesel engine that doesn’t emit lots of NO\textsubscript{x} pollutants.

*Engineering Prowess!*  

*Awesome Engine!*
**Engineering Problem:** Design a diesel engine that doesn’t emit lots of NO\(_x\) pollutants.

**Regulatory Problem:** Design a testing procedure that, given a diesel engine, determines whether it emits lots of NO\(_x\) pollutants.

---

**Engine Testing Regimen**
**Engineering Problem:** Design a diesel engine that doesn’t emit lots of NO$_x$ pollutants.

**Regulatory Problem:** Design a testing procedure that, given a diesel engine, determines whether it emits lots of NO$_x$ pollutants.

**Engineering Prowess!**

**Awesome Engine!**
**Engineering Problem:** Design a diesel engine that doesn’t emit lots of NO\(_x\) pollutants.

**Regulatory Problem:** Design a testing procedure that, given a diesel engine, determines whether it emits lots of NO\(_x\) pollutants.

**Awesome Engine!**

**Engine Testing Regimen**

- **Yep**
- **Nah**
Fact: Almost all “regulatory problems” about computer programs are undecidable. That is, almost all problems of the form “does this program have [behavioral property $X$]” are undecidable.

This can be formalized through a result called **Rice’s Theorem**; take CS154 for details!
Secure Voting

• Suppose that you want to make a voting machine for use in an election between two parties.

• Let $\Sigma = \{r, d\}$. A string $w \in \Sigma^*$ corresponds to a series of votes for the candidates.

• Example: $rrdddrd$ means “two people voted for $r$, then three people voted for $d$, then one more person voted for $r$, then one more person voted for $d$.”
Secure Voting

• A voting machine is a program that takes as input a string of r's and d's, then reports whether person r won the election.

• *Question:* Given a TM that someone claims is a secure voting machine, could we automatically check whether it actually is a secure voting machine?
A secure voting machine is a TM $M$ where $M$ accepts $w \in \{r, d\}^*$ if and only if $w$ has more $r$’s than $d$’s.

```cpp
bool bee(string input) {
    int numRs = countRsIn(input);
    int numDs = countDsIn(input);
    return numRs > numDs;
}
```

A (simple) secure voting machine.

```cpp
bool topaz(string input) {
    return input[0] == 'r';
}
```

A (simple) insecure voting machine.

```cpp
bool anna(string input) {
    int numRs = countRsIn(input);
    int numDs = countDsIn(input);
    if (numRs == numDs) {
        return false;
    } else if (numRs < numDs) {
        return false;
    } else {
        return true;
    }
}
```

An (evil) insecure voting machine.

```cpp
bool green(string input) {
    int n = input.length();
    while (n > 1) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
    }
    int numRs = countRsIn(input);
    int numDs = countDsIn(input);
    return numRs > numDs;
}
```

No one knows!
Secure Voting

- A voting machine is a program that takes as input a string of $r$'s and $d$'s, then reports whether person $r$ won the election.

- **Question**: Given a TM that someone claims is a secure voting machine, could we automatically check whether it actually is a secure voting machine?
A Decider for Secure Voting

• Let’s suppose that, somehow, we managed to build a decider for the secure voting problem.
• Schematically, that decider would look like this:

We could represent this decider in software as a method

```cpp
bool isSecureVotingMachine(string function);
```

that takes as input a function, then returns whether that function is a secure voting machine.
bool isSecureVotingMachine(string function) {
    // Returns whether function accepts only
    // strings with more r's than d's.
}
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (isSecureVotingMachine(me)) {
        return countRsIn(input) <= countDsIn(input);
    } else {
        return countRsIn(input) > countDsIn(input);
    }
}

trickster is a secure voting machine
⇔
isSecureVotingMachine(me) returns true
bool isSecureVotingMachine(string function) {
    // Returns whether function accepts only
    // strings with more r’s than d’s.
}

bool trickster(string input) {
    string me = /* source code of trickster */;
    if (isSecureVotingMachine(me)) {
        return countRsIn(input) <= countDsIn(input);
    } else {
        return countRsIn(input) > countDsIn(input);
    }
}

trickster is a secure voting machine
    ⇐
    isSecureVotingMachine(me) returns true
    ⇐
    trickster isn’t a secure voting machine.
**Theorem:** The secure voting problem is undecidable.

**Proof:** By contradiction; there is a decider $D$ for the secure voting problem. We can represent $D$ as a function

$$
\text{bool isSecureVotingMachine(string function)};
$$

that takes in the source code of a function $function$, then returns whether $function$ is a secure voting machine (that is, whether it accepts precisely the strings with more $r$'s than $d$'s). Given this, consider this function $\text{trickster}$:

```c
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (isSecureVotingMachine(me)) {
        return /* if input has at most as many r’s as d’s */;
    } else {
        return /* if input has more r’s than d’s */;
    }
}
```

Since $\text{isSecureVotingMachine}$ decides the secure voting problem and $me$ holds the source of $\text{trickster}$, we know that

$\text{isSecureVotingMachine}(me)$ returns true if and only if $\text{trickster}$ is a secure voting machine.

Given how $\text{trickster}$ is written, we see that

$\text{isSecureVotingMachine}(me)$ returns true if and only if $\text{trickster}$ isn’t a secure voting machine.

This means that

$\text{trickster}$ is a secure voting machine if and only if $\text{trickster}$ isn’t a secure voting machine.

This is impossible. We’ve reached a contradiction, so our assumption was false and the secure voting problem is undecidable. ■
Interpreting this Result

• The previous argument tells us that there is no general algorithm that we can follow to determine whether a program is a secure voting machine. In other words, any general algorithm to check voting machines will always be wrong on at least one input.

• So what can we do?
  • Design algorithms that work in some, but not all cases. (This is often done in practice.)
  • Fall back on human verification of voting machines. (We do that too.)
  • Carry a healthy degree of skepticism about electronic voting machines. (Then again, did we even need the theoretical result for this?)
Time-Out for Announcements!
Please evaluate this course in Axess.
Your comments really make a difference.
Problem Sets

- PS6 solution released. We are aiming to finish grading by Wednesday noon.
- Problem Set Seven is due on Wednesday at 2:30PM.
Final Exam

• The final exam will be this Saturday from 7-10PM PST at Hewlett 201 (this lecture hall)

• The exam is open-book, open-note, open-internet, and closed-other-humans.
  • Turing Machines will not be covered on this exam.
  • Expect the other topics to scale accordingly.

• You can do this. Best of luck on the exam!
Back to CS103!
Beyond R and RE
Beyond R and RE

• We've now seen how to use self-reference as a tool for showing undecidability (finding languages not in R).

• We still have not broken out of RE yet, though.

• To do so, we will need to build up a better intuition for the class RE.
What exactly is the class RE?
RE, Formally

- Recall that the class **RE** is the class of all recognizable languages:

  \[ \text{RE} = \{ L \mid \text{there is a TM } M \text{ that recognizes } L \} \]

- Since \( \mathbf{R} \neq \text{RE} \), there is no general way to “solve” problems in the class **RE**, if by “solve” you mean “make a computer program that can always tell you the correct answer.”

- So what exactly are the sorts of languages in **RE**?
Does this graph contain four mutually adjacent nodes?
Does this graph contain four mutually adjacent nodes?
Does this graph contain four mutually adjacent nodes?
Key Intuition:

A language $L$ is in RE if, for any string $w$, if you are convinced that $w \in L$, there is some way you could prove that to someone else.
Verification

11

Does the hailstone sequence terminate for this number?
Verification

11

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

34

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

17

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

52

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

26

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

13

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

40

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

20

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

10

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

16

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

8

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

2

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

1

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

11

Does the hailstone sequence terminate for this number?
Verification

11

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

34

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

17

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

52

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

26

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verification

13

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?
Verifiers

- A **verifier** for a language $L$ is a TM $V$ with the following two properties:
  
  $V$ halts on all inputs.

  $\forall w \in \Sigma^*. (w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle)$

- Intuitively, what does this mean?
Deciders and Verifiers

Decider $M$ for $L$

- **Solve the problem**
- $M$ halts on all inputs.
- $w \in L \iff M$ accepts $w$

Verifier $V$ for $L$

- **Check an answer**
- $V$ halts on all inputs.
- $w \in L \iff \exists c \in \Sigma^*. V$ accepts $\langle w, c \rangle$

- If $M$ accepts, then $w \in L$. (Yes!)
- If $M$ rejects, then $w \not\in L$. (No!)
- If $V$ accepts $\langle w, c \rangle$, then $w \in L$. (Yes!)
- If $V$ rejects $\langle w, c \rangle$, we don't know whether $w \in L$. (Not sure)
Verifiers

• A **verifier** for a language $L$ is a TM $V$ with the following properties:

  $$V \text{ halts on all inputs.}$$

  $$\forall w \in \Sigma^*. \ (w \in L \leftrightarrow \exists c \in \Sigma^*. \ V \text{ accepts } (w, c))$$

• Some notes about $V$:
  
  • If $V$ accepts $(w, c)$, we're guaranteed $w \in L$.
  
  • If $V$ rejects $(w, c)$, then either
    
    - $w \in L$, but you gave the wrong $c$, or
    
    - $w \notin L$, so no possible $c$ will work.
Verifiers

- A **verifier** for a language $L$ is a TM $V$ with the following properties:

  $$V \text{ halts on all inputs.}$$

  $$\forall w \in \Sigma^*. (w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } (w, c))$$

- Some notes about $V$:
  - Notice that the certificate $c$ is existentially quantified. Any string $w \in L$ must have at least one $c$ that causes $V$ to accept, and possibly more.
  - $V$ is required to halt, so given any potential certificate $c$ for $w$, you can check whether the certificate is correct.
Verifiers

A **verifier** for a language $L$ is a TM $V$ with the following properties:

- **$V$ halts on all inputs.**
- $\forall w \in \Sigma^*. (w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle)$

Some notes about $V$:

- Notice that $V$ isn’t a decider for $L$ and isn’t a recognizer for $L$.
- The job of $V$ is just to check certificates, not to decide membership in $L$. 
Verifiers

- A **verifier** for a language $L$ is a TM $V$ with the following properties:
  
  **V halts on all inputs.**
  
  $\forall w \in \Sigma^*. \ (w \in L \leftrightarrow \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle)$

- Some notes about $V$:
  
  - Although this formal definition works with a string $c$, remember that $c$ can be an encoding of some other object.
  
  - In practice, $c$ will likely just be “some other auxiliary data that helps you out.”
A Very Nifty Verifier

- Consider \( A_{TM} \):

\[
A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}.
\]

- This is a *canonical* example of an undecidable language. There’s no way, in general, to tell whether a TM \( M \) will accept a string \( w \).

- Although this language is undecidable, it’s an RE language, and it’s possible to build a verifier for it!
A Very Nifty Verifier

- Consider $A_{TM}$:
  
  $$A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}.$$  

- We know that $U_{TM}$ is a recognizer for $A_{TM}$. It is also a verifier for $A_{TM}$?

- No, for two reasons:
  
  - $U_{TM}$ doesn’t always halt. 
    
    *Do you see why?*
  
  - $U_{TM}$ takes as input a TM $M$ and a string $w$. A verifier also needs a certificate.
A Very Nifty Verifier

- Consider $A_{\text{TM}}$:
  $$A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}. $$

- A verifier for $A_{\text{TM}}$ would take as input:
  - A TM $M$,
  - a string $w$, and
  - a certificate $c$.

- The certificate $c$ should be some evidence that suggests that $M$ accepts $w$.

- What could our certificate be?
Some Verifiers

- Consider $A_{\text{TM}}$:

$$A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$ 

```java
bool checkWillAccept(TM M, string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on w;
    }
    return whether M is in an accepting state;
}
```

- Do you see why $M$ accepts $w$ if and only if there is a $c$ such that `checkWillAccept(M, w, c)` returns true?

- Do you see why `checkWillAccept` always halts?
What languages are verifiable?
**Theorem:** If $L$ is a language, then there is a verifier for $L$ if and only if $L \in \text{RE}$.
Where We’ve Been

- NFA
- Regex
- State Elimination
- Thompson’s Algorithm
Where We’re Going

Verifier

Try all certificates

Recognizer

Enforce a step count
**Theorem:** If there is a verifier \( V \) for a language \( L \), then \( L \in \text{RE} \).

**Proof goal:** Given a verifier \( V \) for a language \( L \), find a way to construct a recognizer \( M \) for \( L \).
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier \( V \) for a language \( L \), then \( L \in \text{RE} \).

- **Proof goal:** Given a verifier \( V \) for a language \( L \), find a way to construct a recognizer \( M \) for \( L \).
Verifiers and $\textbf{RE}$

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

```
Verifier V for L
```

```
input string (w)
certificate (c)
```

```
"Check the answer"
```

```
yes!
not sure
```

```
ε a b aa ab ba bb aaa aab aba abb baa...
```
Verifiers and $\textbf{RE}$

- **Theorem**: If there is a verifier $V$ for a language $L$, then $L \in \textbf{RE}$.
- **Proof goal**: Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$.
Verifiers and $\textbf{RE}$

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \textbf{RE}$.
- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

![Diagram of verifier and recognizer process]
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.
- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

Diagram:

```
\begin{figure}
\centering
\begin{tikzpicture}
  \node (input) at (0,0) {	extit{input string (w)}};
  \node (certificate) at (0,-1) {	extit{certificate (c)}};
  \node (verifier) at (2,0) {Verifier $V$ for $L$};
  \node (yes) at (4,0) {yes!};
  \node (not_sure) at (4,-1) {not sure};

  \draw[->] (input) -- (verifier);
  \draw[->] (certificate) -- (verifier);
  \draw[->] (verifier) -- (yes);
  \draw[->] (verifier) -- (not_sure);

  \end{tikzpicture}
\end{figure}
```
Verifiers and $\textbf{RE}$

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \textbf{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$.
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \textbf{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

![Diagram of verifier process]

- Input string $(w)$
- Certificate $(c)$
- “Check the answer”
- Yes!
- Not sure
• **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

• **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

Verifiers and $\text{RE}$
Verifiers and RE

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

![Diagram showing the interaction between input string and certificate, and the outcomes of verification.]
Verifiers and \textbf{RE}

- \textbf{Theorem}: If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- \textbf{Proof goal}: Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

\begin{itemize}
  \item \textbf{Verifier $V$ for $L$}:
    \begin{itemize}
      \item \textit{input string} $(w)$
      \item \textit{certificate} $(c)$
      \end{itemize}
  \end{itemize}

  \begin{align*}
  &\text{"Check the answer"} & & \text{yes!} \\
  &\text{not sure} \\
  &\varepsilon & a & b & aa & ab & ba & bb & aaa & aab & aba & abb & baa & \ldots
  \end{align*}
Verifiers and RE

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

**Diagram:**

- **Verifier V for L**
  - Input string $(w)$
  - Certificate $(c)$
  - “Check the answer”
  - Yes!
  - Not sure

**Strings:**

\[ \varepsilon, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots \]
Verifiers and **RE**

- **Theorem**: If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- **Proof goal**: Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$.

```
\begin{array}{l}
\text{input string (w)} \\
\text{certificate (c)}
\end{array}
\Rightarrow
\text{Verifier V for L}
\Rightarrow
\text{"Check the answer"}
\Rightarrow
\begin{cases} yes! \\ not sure \end{cases}
\Rightarrow
\begin{array}{l}
\varepsilon \ a \ b \ aa \ ab \ ba \ bb \ aaa \ aab \ aba \ abb \ baa \ ...
\end{array}
```
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \textbf{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

\[ \epsilon \ a \ b \ aa \ ab \ ba \ bb \ aaa \ aab \ aba \ abb \ baa \ldots \]
Verifiers and RE

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\[
\begin{array}{c}
\varepsilon \\
a \\
b \\
\text{aa} \\
\text{ab} \\
\text{ba} \\
\text{bb} \\
\text{aaa} \\
\text{aab} \\
\text{aba} \\
\text{abb} \\
\text{baa} \\
\text{...}
\end{array}
\]
Verifiers and \textbf{RE}

- \textbf{Theorem:} If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- \textbf{Proof goal:} Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

\begin{itemize}
  \item \textbf{Verifier $V$ for $L$} \hfill \text{"Check the answer"} \hfill \text{yes!}
  \item \textbf{Input string ($w$)} \hfill \text{not sure}
  \item \textbf{Certificate ($c$)}
\end{itemize}

\begin{itemize}
  \item $\varepsilon$
  \item a
  \item b
  \item aa
  \item ab
  \item ba
  \item bb
  \item aaa
  \item aab
  \item aba
  \item abb
  \item baa
  \item ...
\end{itemize}
Verifiers and RE

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Verifiers and \textbf{RE}

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\[\begin{array}{c}
epsilon \\
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\ldots \\
\end{array}\]
Verifiers and \textbf{RE}

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```
\begin{tabular}{c}
\varepsilon \\
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\ldots
\end{tabular}
```
Verifiers and \textbf{RE}

- \textbf{Theorem:} If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.
- \textbf{Proof goal:} Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 

```
\begin{center}
\begin{tikzpicture}
  \node (input) {input string ($w$)};
  \node (certificate) {certificate ($c$)};
  \node (verifier) {Verifier $V$ for $L$} [right of=input, below of=certificate, draw, fill=blue!20];
  \node (check) {"Check the answer"} [right of=verifier];
  \node (yes) [right of=check, draw, fill=green!20] {yes!} [above of=check];
  \node (not sure) [right of=check, draw, fill=orange!20] {not sure} [below of=check];

  \draw[->] (input) -- (verifier);
  \draw[->] (input) -- (check);
  \draw[->] (certificate) -- (verifier);
  \draw[->] (certificate) -- (check);
  \draw[->] (verifier) -- (check);
  \draw[->] (check) -- (yes);
  \draw[->] (check) -- (not sure);
\end{tikzpicture}
\end{center}
```
Verifiers and \textbf{RE}

- \textbf{Theorem:} If there is a verifier \( V \) for a language \( L \), then \( L \in \text{RE} \).

- \textbf{Proof goal:} Given a verifier \( V \) for a language \( L \), find a way to construct a recognizer \( M \) for \( L \).

\[ \varepsilon \quad a \quad b \quad aa \quad ab \quad ba \quad bb \quad aaa \quad aab \quad aba \quad abb \quad baa \quad \ldots \]
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$.

\[
\begin{array}{c}
\text{Verifier V for } L \\
\text{:} & \text{``Check the answer''}
\end{array}
\]

\[
\begin{array}{c}
\text{input string (w)} \\
\vdash & \\
\text{certificate (c)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{yes!} \\
\not \text{ sure}
\end{array}
\]

\[
\begin{array}{c}
\varepsilon & a & b & aa & ab & ba & bb & aaa & aab & aba & abb & baa & \ldots
\end{array}
\]
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

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\begin{array}{c}
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\text{b} \\
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\text{aaa} \\
\text{aab} \\
\text{aba} \\
\text{abb} \\
\text{baa} \\
\vdots
\end{array}
\]
Verifiers and \textbf{RE}

- \textbf{Theorem}: If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

- \textbf{Proof goal}: Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$.

---

***Verifier V for L***

- **Input string** ($w$)
- **Certificate** ($c$)

---

"Check the answer"

- \textit{yes!}
- \textit{not sure}

---

\begin{align*}
\epsilon & \quad a & \quad b & \quad aa & \quad ab & \quad ba & \quad bb & \quad aaa & \quad aab & \quad aba & \quad abb & \quad baa & \quad ...
\end{align*}
**Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

**Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$. 
Verifiers and \textbf{RE}

\begin{itemize}
\item \textbf{Theorem:} If $V$ is a verifier for $L$, then $L \in \textbf{RE}$.
\item \textbf{Proof sketch:} Consider the following program:
\begin{verbatim}
bool isInL(string w) {
    for (each string c) {
        if (V accepts $\langle w, c \rangle$) return true;
    }
}
\end{verbatim}
If $w \in L$, there is some $c \in \Sigma^*$ where $V$ accepts $\langle w, c \rangle$. The function \texttt{isInL} tries all possible strings as certificates, so it will eventually find $c$ (or some other working certificate), see $V$ accept $\langle w, c \rangle$, then return true. Conversely, if \texttt{isInL}($w$) returns true, then there was some string $c$ such that $V$ accepted $\langle w, c \rangle$, so we see that $w \in L$. ■
\end{itemize}
Verifiers and RE

- **Theorem:** If $L \in \mathbf{RE}$, then there is a verifier for $L$.
- **Proof goal:** Beginning with a recognizer $M$ for the language $L$, show how to construct a verifier $V$ for $L$. 
We have a recognizer for a language. We want to turn it into a verifier. Where did we see this before?
Some Verifiers

Consider $A_{TM}$:

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$ 

```cpp
bool checkWillAccept(TM M, string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on w;
    }
    return whether M is in an accepting state;
}
```

Observation: This trick of enforcing a step count limits how long $M$ can run for!

Do you see why $M$ accepts $w$ iff there is some $c$ such that checkWillAccept($M, w, c$) returns true?
Do you see why checkWillAccept always halts?
Verifiers and \textbf{RE}

- \textbf{Theorem:} If $L \in \textbf{RE}$, then there is a verifier for $L$.

- \textbf{Proof sketch:} Let $L$ be a \textbf{RE} language and let $M$ be a recognizer for it. Consider this function:

```cpp
bool checkIsInL(string w, int c) {
    TM M = /* hardcoded version of a recognizer for L */;
    set up a simulation of $M$ running on $w$;
    for (int i = 0; i < c; i++) {
        simulate the next step of $M$ running on $w$;
    }
    return whether $M$ is in an accepting state;
}
```

Note that \texttt{checkIsInL} always halts, since each step takes only finite time to complete. Next, notice that if there is a $c$ where \texttt{checkIsInL}(w, c) returns true, then $M$ accepted $w$ after running for $c$ steps, so $w \in L$. Conversely, if $w \in L$, then $M$ accepts $w$ after some number of steps (call that number $c$). Then \texttt{checkIsInL}(w, c) will run $M$ on $w$ for $c$ steps, watch $M$ accept $w$, then return true. ■
**RE and Proofs**

- Verifiers and recognizers give two different perspectives on the “proof” intuition for RE.
- Verifiers are explicitly built to check proofs that strings are in the language.
  - If you know that some string $w$ belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.
- You can think of a recognizer as a device that “searches” for a proof that $w \in L$.
  - If it finds it, great!
  - If not, it might loop forever.
RE and Proofs

- If the RE languages represent languages where membership can be proven, what does a non-RE language look like?
- Intuitively, a language is not in RE if there is no general way to prove that a given string $w \in L$ actually belongs to $L$.
- In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!
Finding Non-**RE** Languages
Finding Non-RE Languages

- Right now, we know that non-RE languages exist, but we have no idea what they look like.
- How might we find one?
Recognizers and Recognizability

- **Recall:** We say that $M$ is a recognizer for $L$ if the following is true:

  $$\forall w \in \Sigma^*. (w \in L \leftrightarrow M \text{ accepts } w).$$

- This above description applies to all strings, including strings that, by pure coincidence, happen to be encodings of TMs.

- What happens if we list off all Turing machines, looking at how those TMs behave given other TMs as input?
All Turing machines, listed in some order.
All descriptions of TMs, listed in the same order.
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\{ $\langle M \rangle \mid M$ is a TM that does not accept $\langle M \rangle$ \}
Diagonalization Revisited

• The *diagonalization language*, which we denote $L_D$, is defined as

\[
L_D = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ does not accept } \langle M \rangle \}
\]

• We constructed this language to be different from the language of every TM.

• Therefore, $L_D \notin \text{RE}$! Let’s go prove this.
\[L_D = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ does not accept } \langle M \rangle \}\]

**Theorem:** \(L_D \notin \text{RE}\.\)

**Proof:** Assume for the sake of contradiction that \(L_D \in \text{RE}\.\) This means that there is a recognizer \(R\) for \(L_D\.\)

Now, focus on what happens if we run recognizer \(R\) on its own string encoding (that is, running \(R\) on \(\langle R \rangle\)). Since \(R\) is a recognizer for \(L_D\), we see that

\[R \text{ accepts } \langle R \rangle \quad \text{if and only if} \quad \langle R \rangle \in L_D.\]

By definition of \(L_D\), we know that

\[\langle R \rangle \in L_D \quad \text{if and only if} \quad R \text{ does not accept } \langle R \rangle.\]

Combining the two above statements tells us that

\[R \text{ accepts } \langle R \rangle \quad \text{if and only if} \quad R \text{ does not accept } \langle R \rangle.\]

This is impossible. We’ve reached a contradiction, so our assumption was wrong, and so \(L_D \notin \text{RE}\.\) \(\blacksquare\)
What This Means

● On a deeper philosophical level, the fact that non-RE languages exist supports the following claim:

There are statements that are true but not provable.

● Intuitively, given any non-RE language, there will be some string in the language that cannot be proven to be in the language.

● This result can be formalized as a result called Gödel's incompleteness theorem, one of the most important mathematical results of all time.

● Want to learn more? Take Phil 152 or CS154!
What This Means

• On a more philosophical note, you could interpret the previous result in the following way:

  *There are inherent limits about what mathematics can teach us.*

• There's no automatic way to do math. There are true statements that we can't prove.

• That doesn't mean that mathematics is worthless. It just means that we need to temper our expectations about it.
There are more problems to solve than there are programs capable of solving them.
There is so much more to explore and so many big questions to ask – many of which haven't been asked yet!
Our questions to you:

What problems will you *choose* to solve? Why do those problems matter to you? And how are you going to solve them?