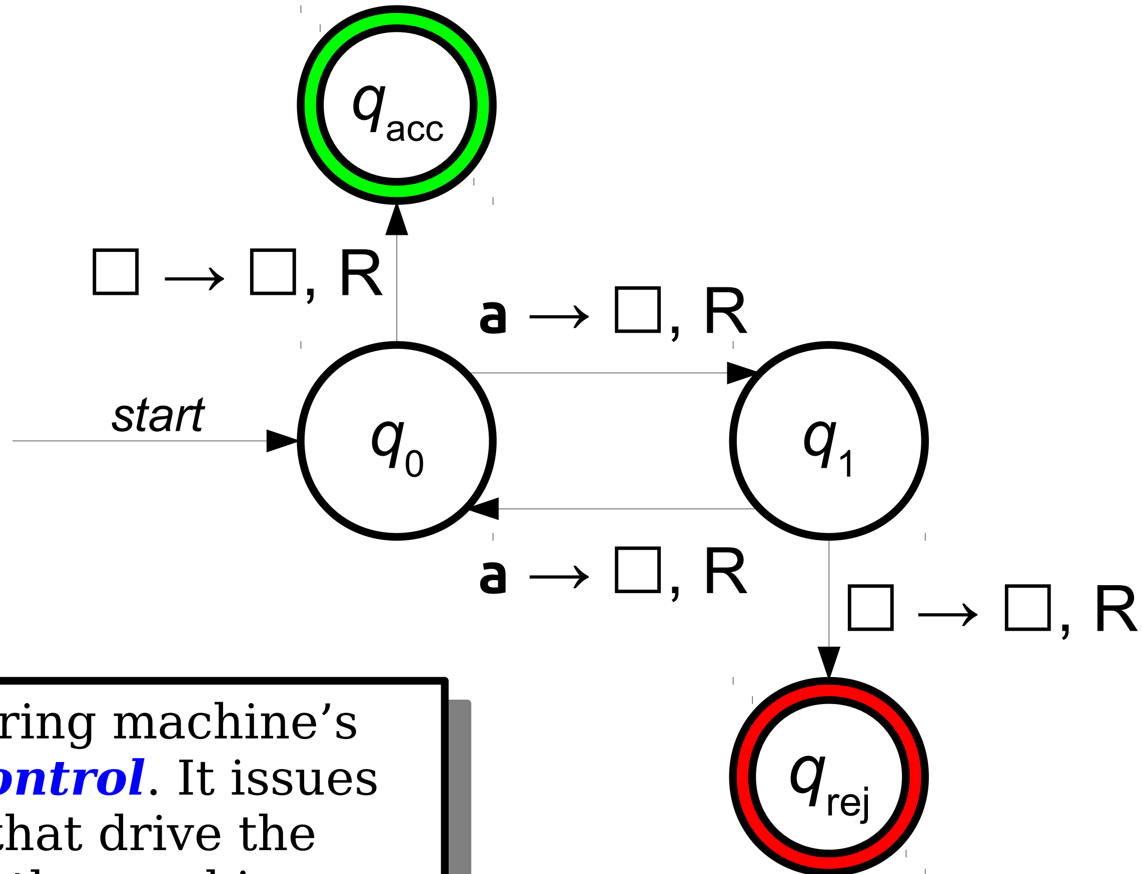
Turing Machines

Part Two

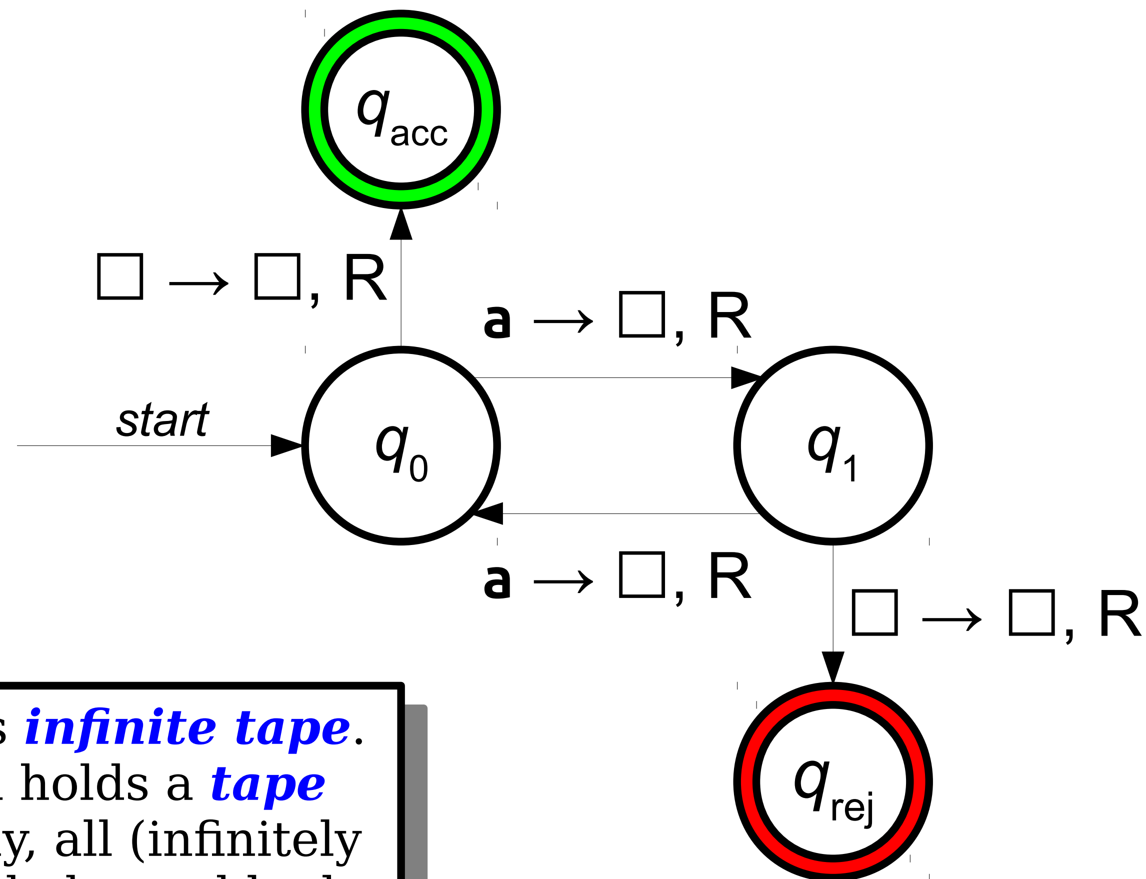
Recap from Last Time

Our First Turing Machine

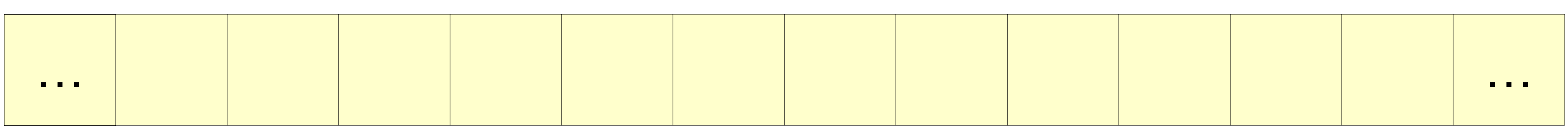


This is the Turing machine's ***finite state control***. It issues commands that drive the operation of the machine.

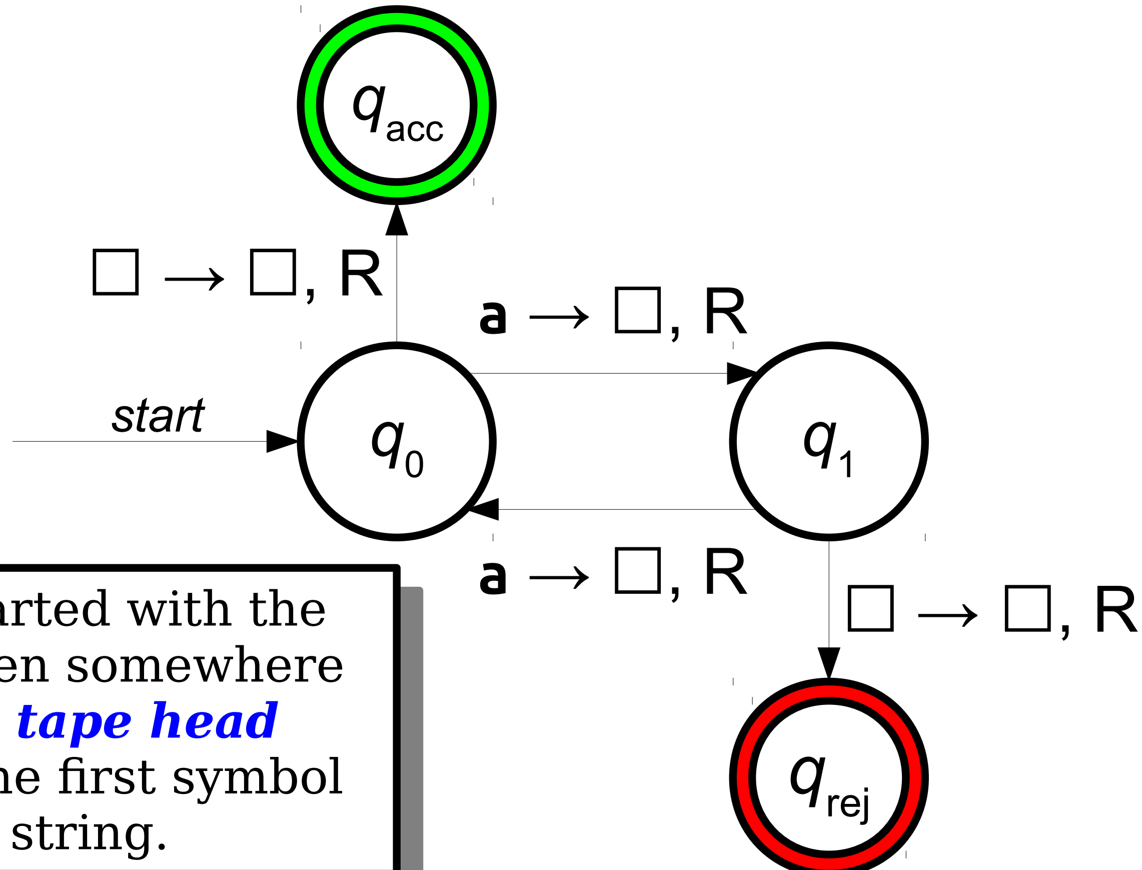
Our First Turing Machine



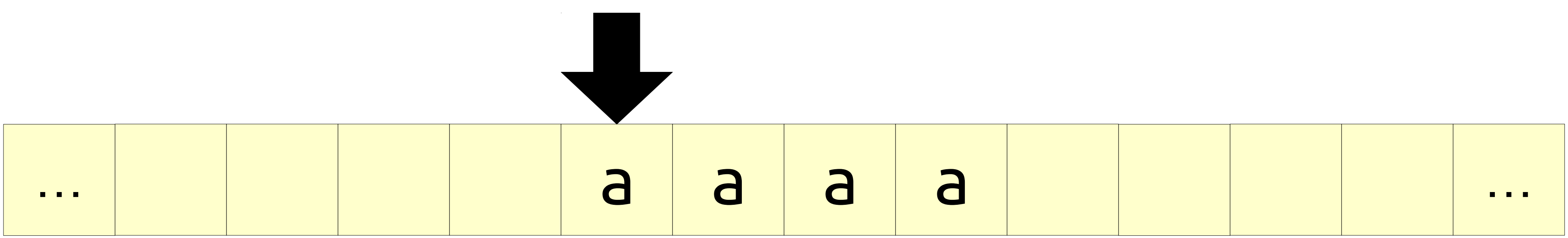
This is the TM's ***infinite tape***. Each tape cell holds a ***tape symbol***. Initially, all (infinitely many) tape symbols are blank.



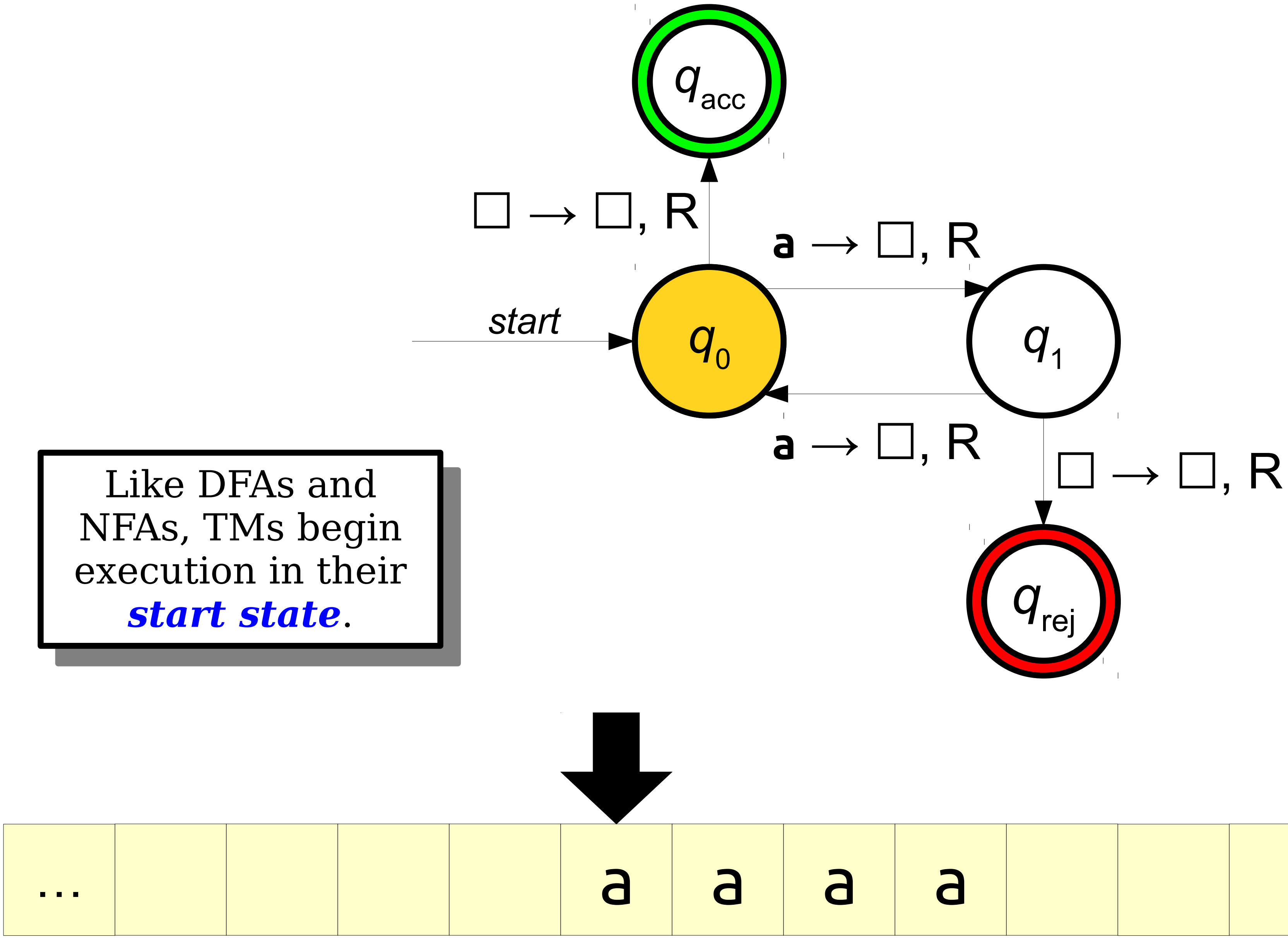
Our First Turing Machine



The machine is started with the ***input string*** written somewhere on the tape. The ***tape head*** initially points to the first symbol of the input string.

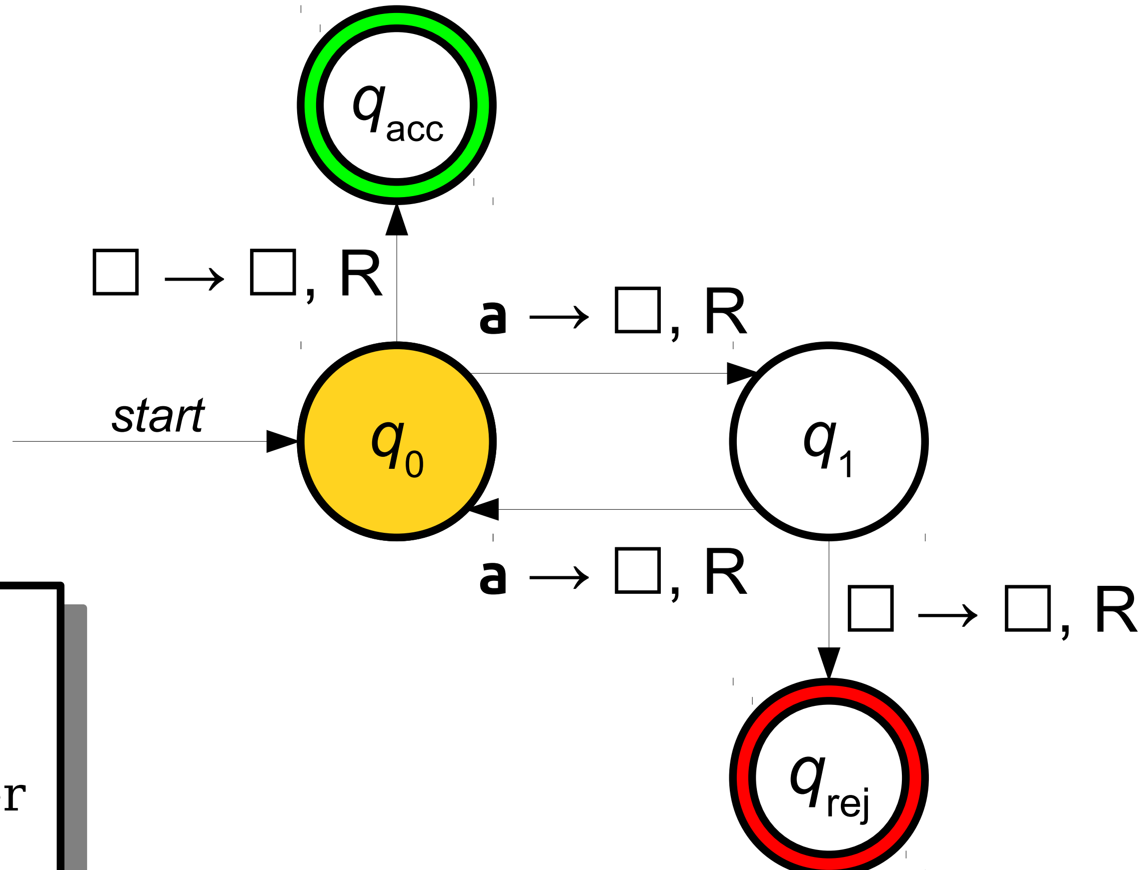


Our First Turing Machine



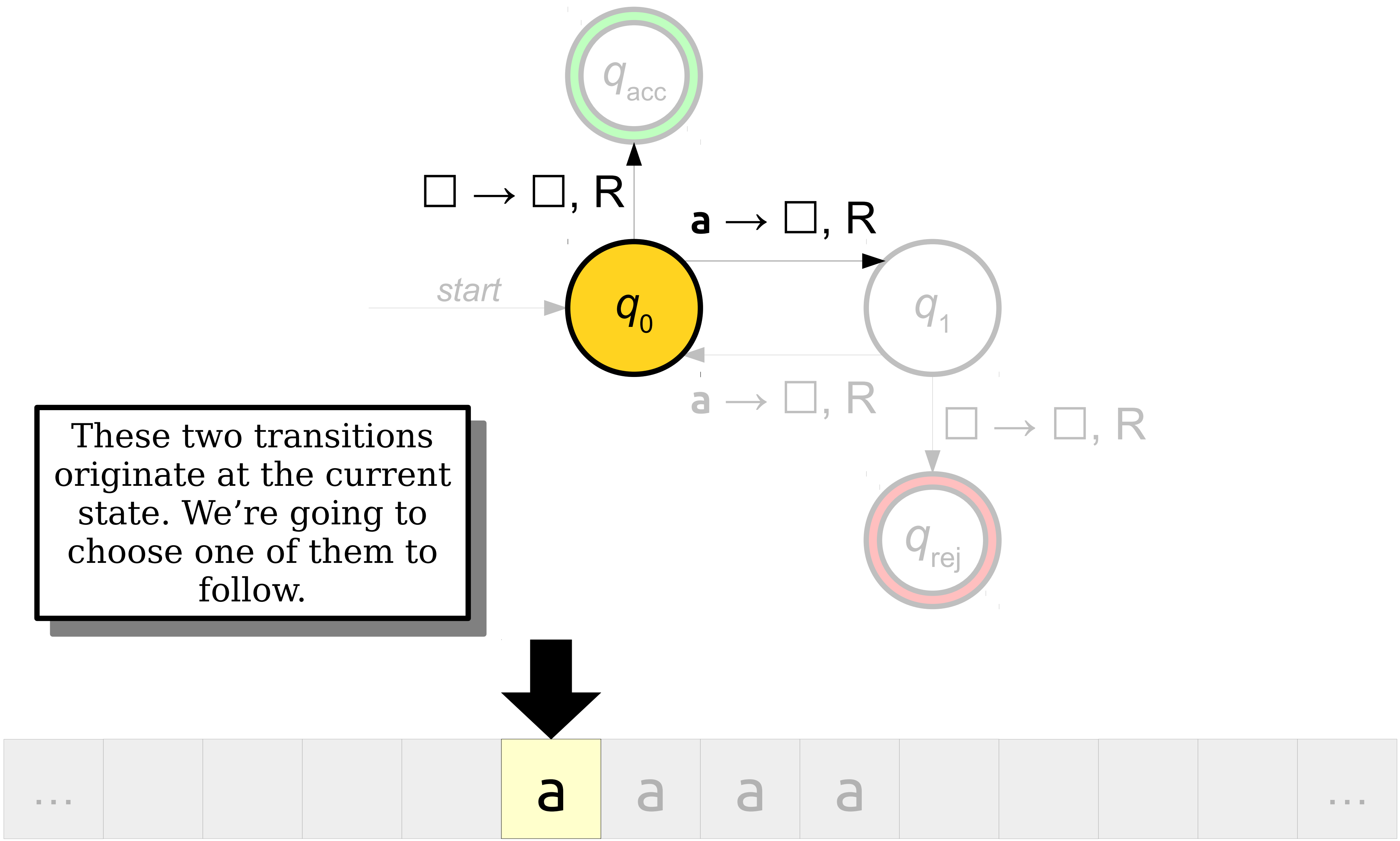
Like DFAs and NFAs, TMs begin execution in their ***start state***.

Our First Turing Machine



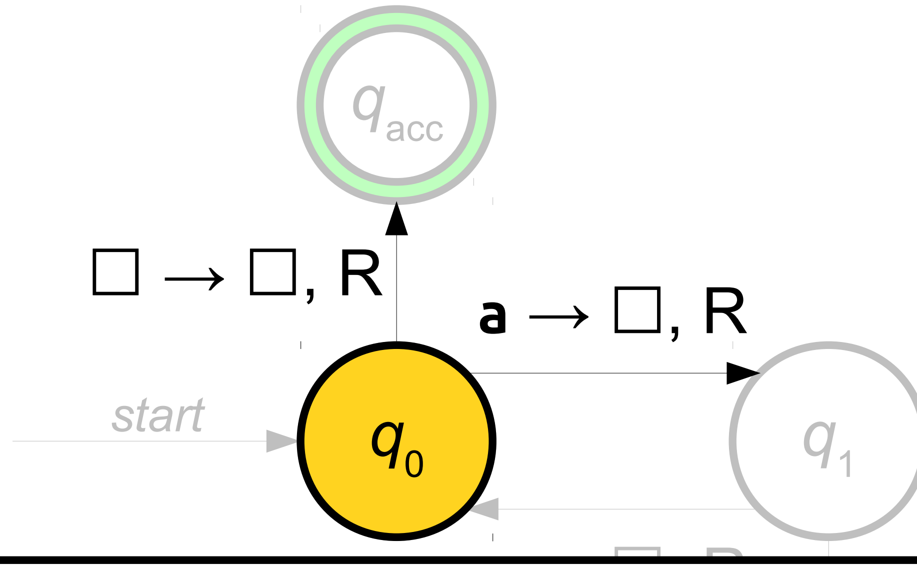
At each step, the TM only looks at the symbol immediately under the ***tape head***.

Our First Turing Machine



These two transitions originate at the current state. We're going to choose one of them to follow.

Our First Turing Machine

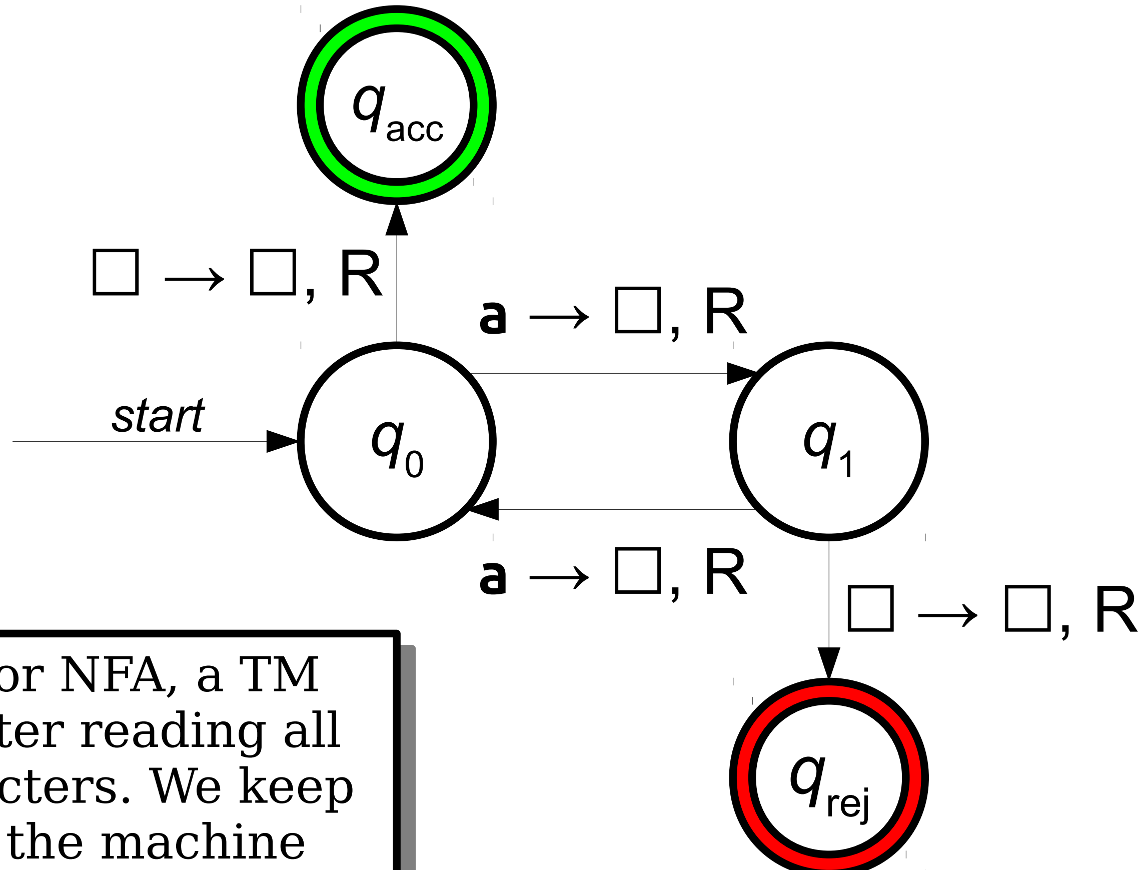


Each transition has the form

read → write, dir

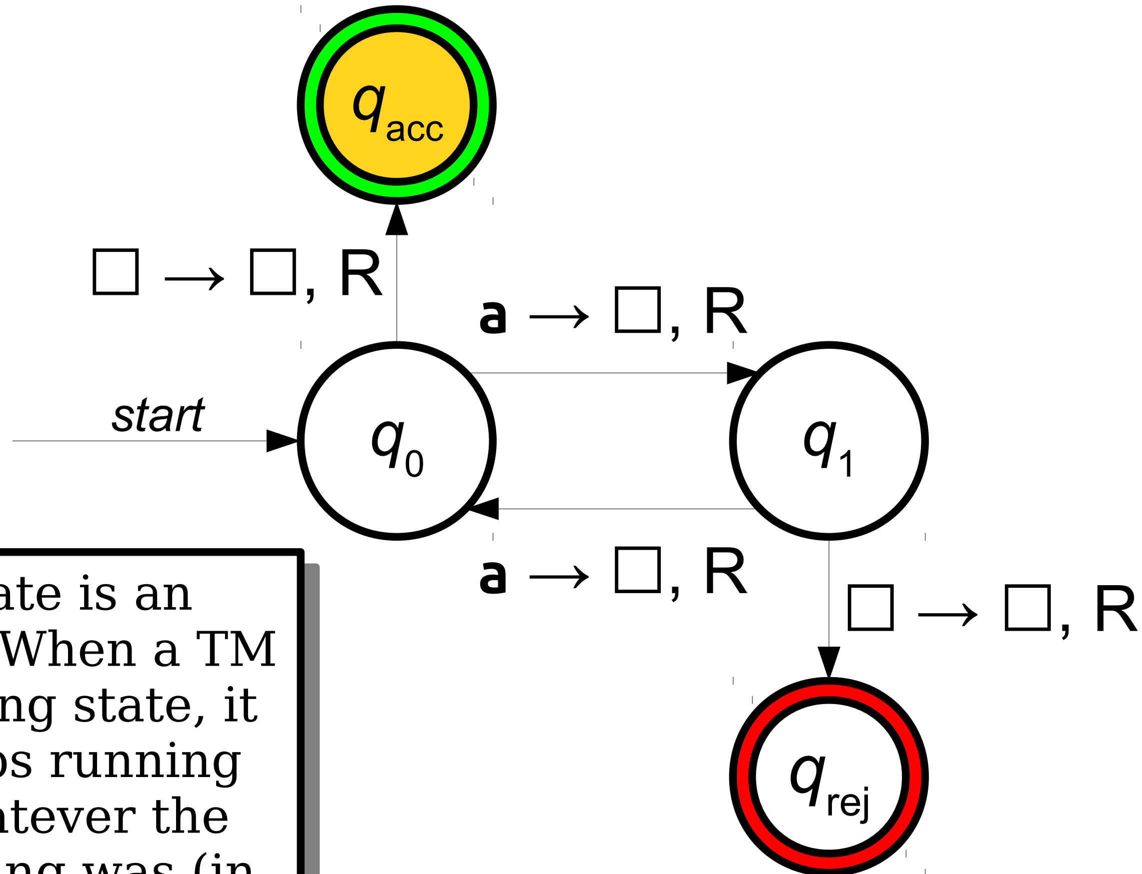
and means "if symbol ***read*** is under the tape head, replace it with ***write*** and move the tape head in direction ***dir*** (L or R). The □ symbol denotes a blank cell.

Our First Turing Machine



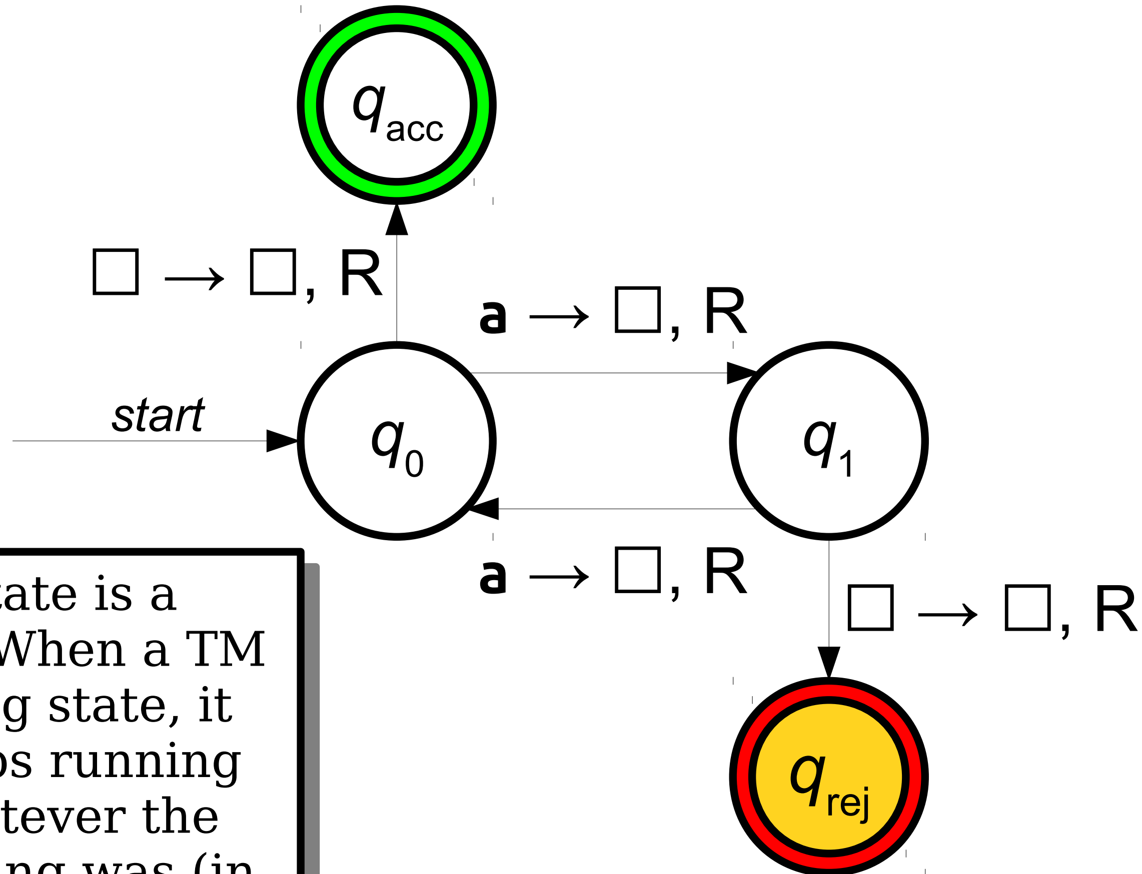
Unlike a DFA or NFA, a TM doesn't stop after reading all the input characters. We keep running until the machine explicitly says to stop.

Our First Turing Machine



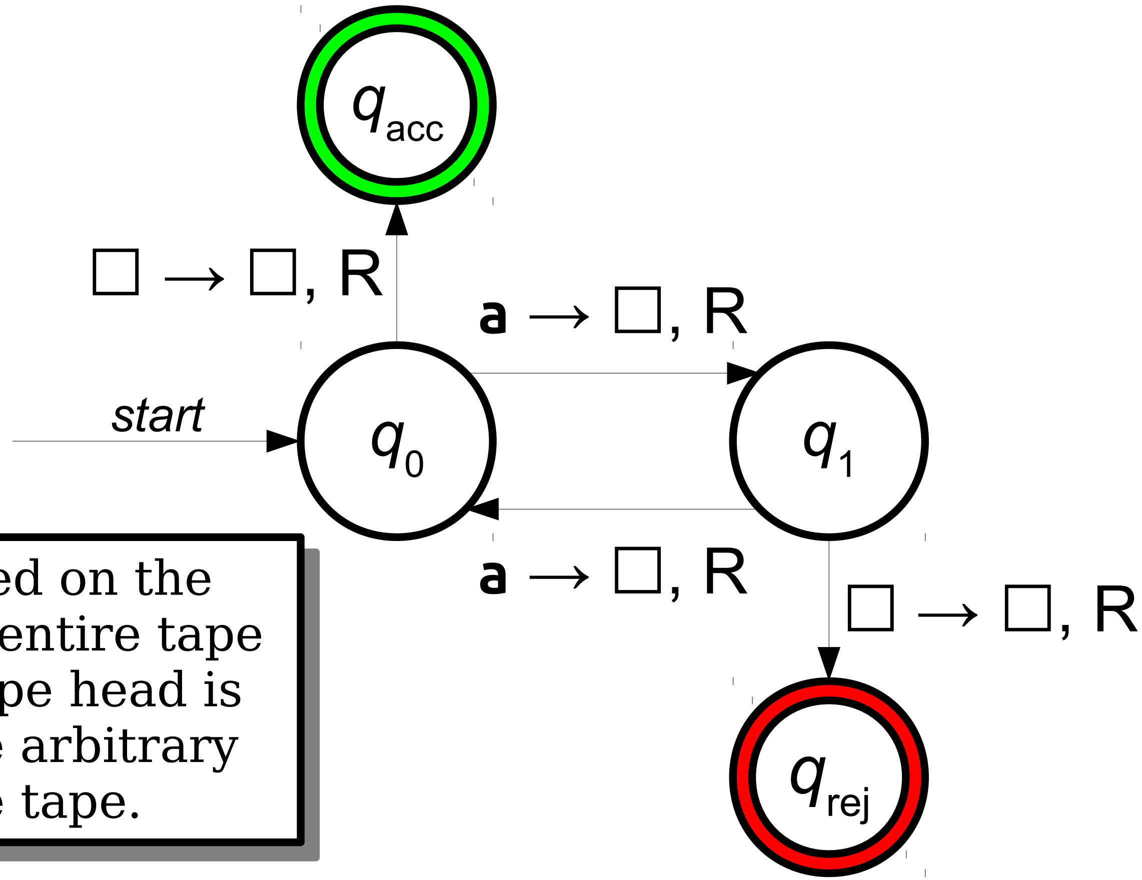
This special state is an ***accepting state***. When a TM enters an accepting state, it *immediately* stops running and accepts whatever the original input string was (in this case, **aaaa**).

Our First Turing Machine

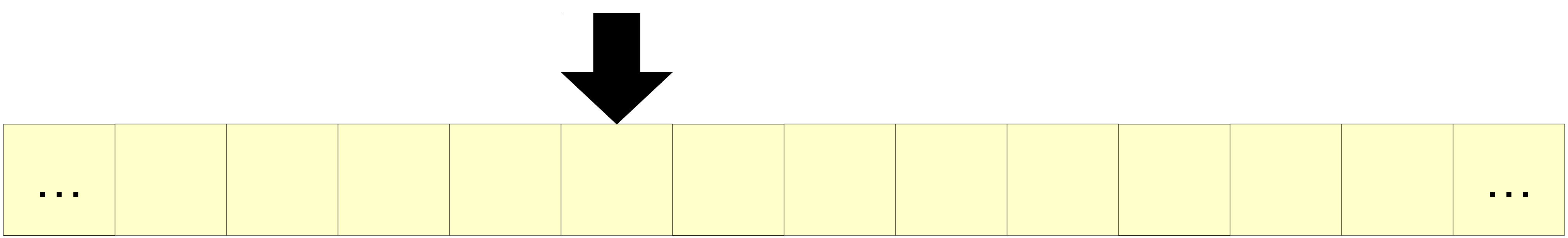


This special state is a ***rejecting state***. When a TM enters a rejecting state, it *immediately* stops running and rejects whatever the original input string was (in this case, **aaaaa**).

Our First Turing Machine

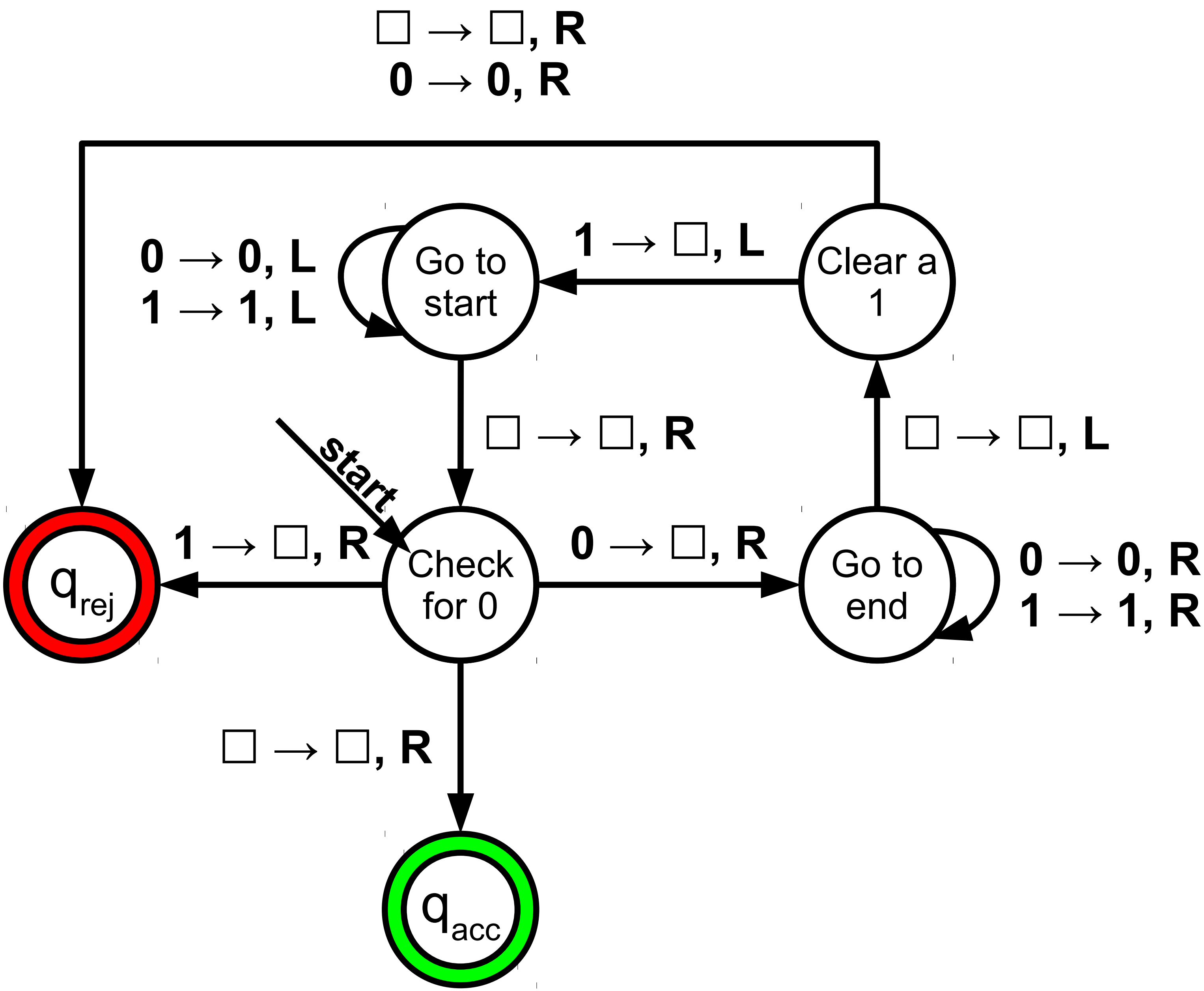


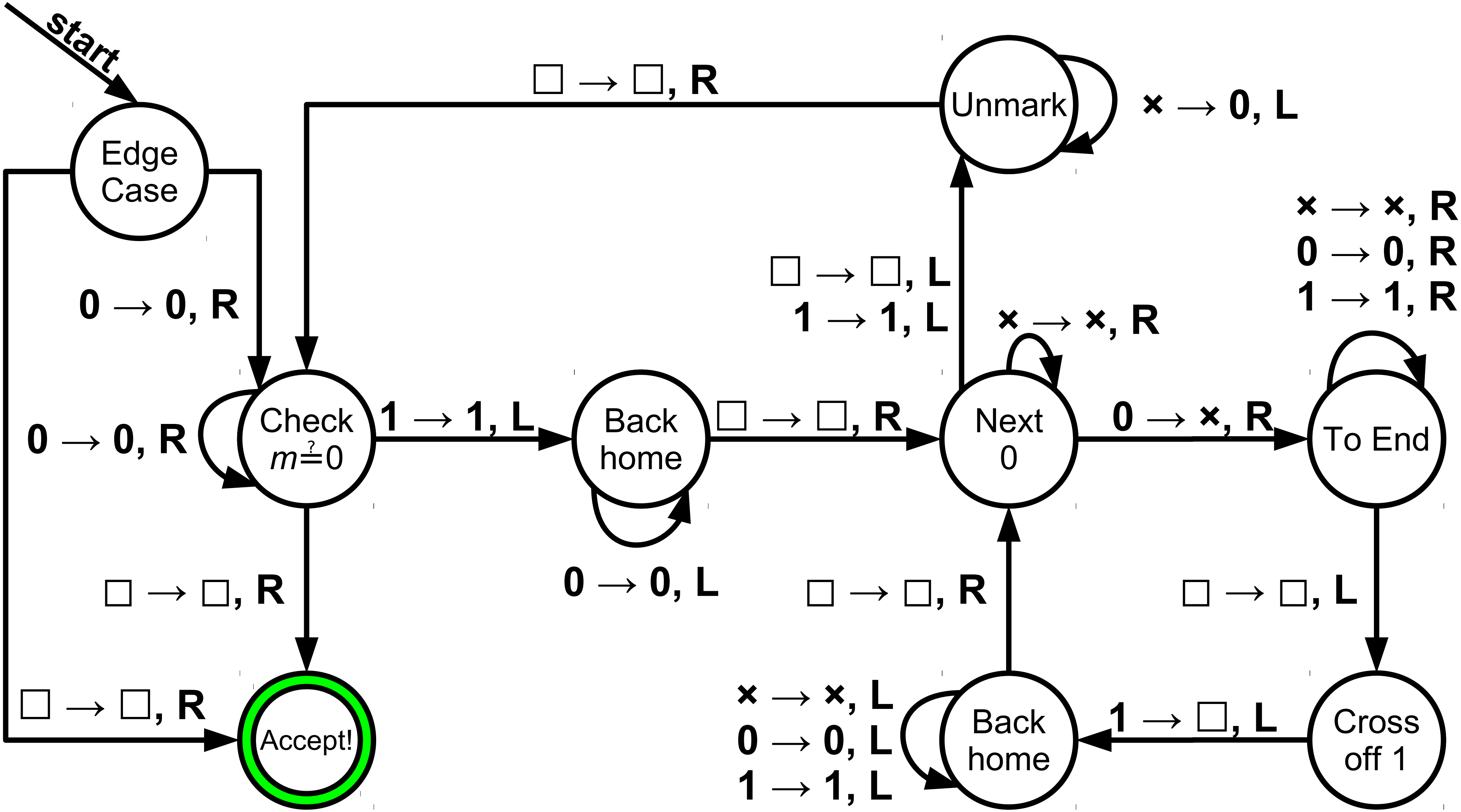
If the TM is started on the empty string ε, the entire tape is blank and the tape head is positioned at some arbitrary location on the tape.



Input and Tape Alphabets

- A Turing machine has two alphabets:
  - An ***input alphabet*** Σ. All input strings are written in the input alphabet.
  - A ***tape alphabet*** Γ, where $\Sigma \subseteq \Gamma$. The tape alphabet contains all symbols that can be written onto the tape.
- The tape alphabet Γ can contain any number of symbols, but always contains at least one ***blank symbol***, denoted $\square$. You are guaranteed $\square \notin \Sigma$.
- At startup, the Turing machine begins with an infinite tape of $\square$ symbols with the input written at some location. The tape head is positioned at the start of the input.


□ → □, R
0 → 0, R
0 → 0, L
1 → 1, L
Go to start
1 → □, L
Clear a 1
□ → □, R
□ → □, L
start
q_{rej}
1 → □, R
Check for 0
0 → □, R
Go to end
0 → 0, R
1 → 1, R
□ → □, R
q_{acc}


start
Edge Case
Check $m \stackrel{?}{=} 0$
Back home
Next 0
To End
Unmark
Cross off 1
Back home
Accept!
0 → 0, R
0 → 0, R
□ → □, R
□ → □, R
1 → 1, L
0 → 0, L
□ → □, R
□ → □, R
□ → □, L
1 → 1, L
× → ×, R
× → 0, L
0 → ×, R
× → ×, R
0 → 0, R
1 → 1, R
□ → □, L
1 → □, L
□ → □, R
× → ×, L
0 → 0, L
1 → 1, L

New Stuff!

Main Question for Today:

Just how powerful are Turing machines?

Another TM Design

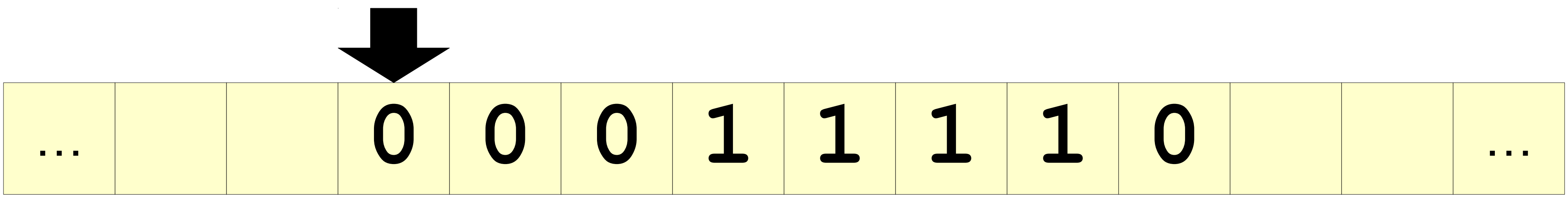
- Last time, we designed a TM for this language over $\Sigma = \{\mathtt{0}, \mathtt{1}\}$:

  $L = \{ w \in \Sigma^* \mid w$ has the same number of $\mathtt{0}$s and $\mathtt{1}$s $\}$

- Let's do a quick review of how it worked.

A Different Idea

A Different Strategy

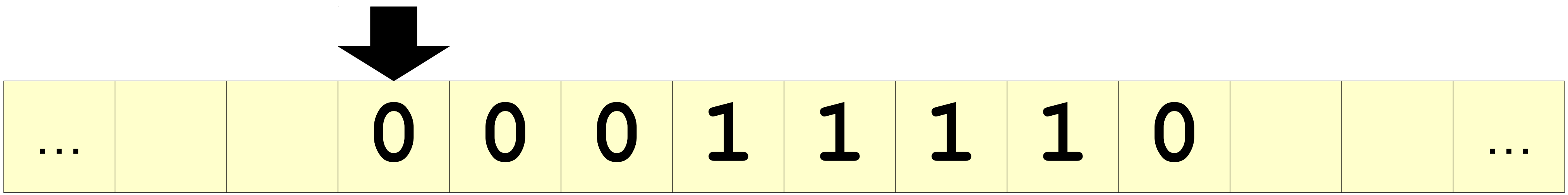


Last time, we built a machine that checks whether a string has the form 0^n1^n.

That machine almost solves this problem, but requires that the characters have to be in order.

What if we sorted the input?

A Different Strategy



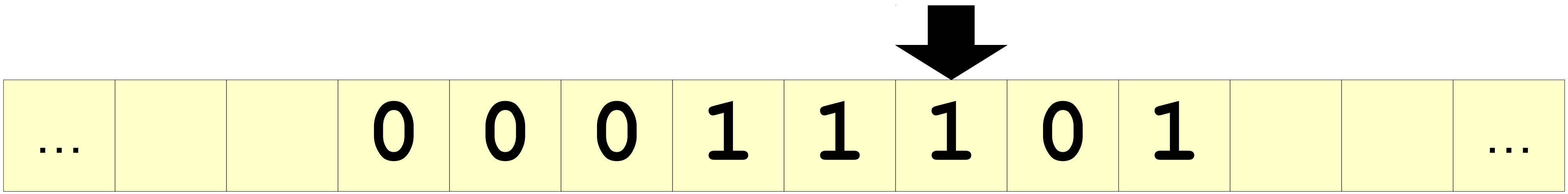
Observation 1: A string of 0s and 1s is sorted if it matches the regex 0*1*.

A Different Strategy

…			0	0	0	1	1	1	1	0			…

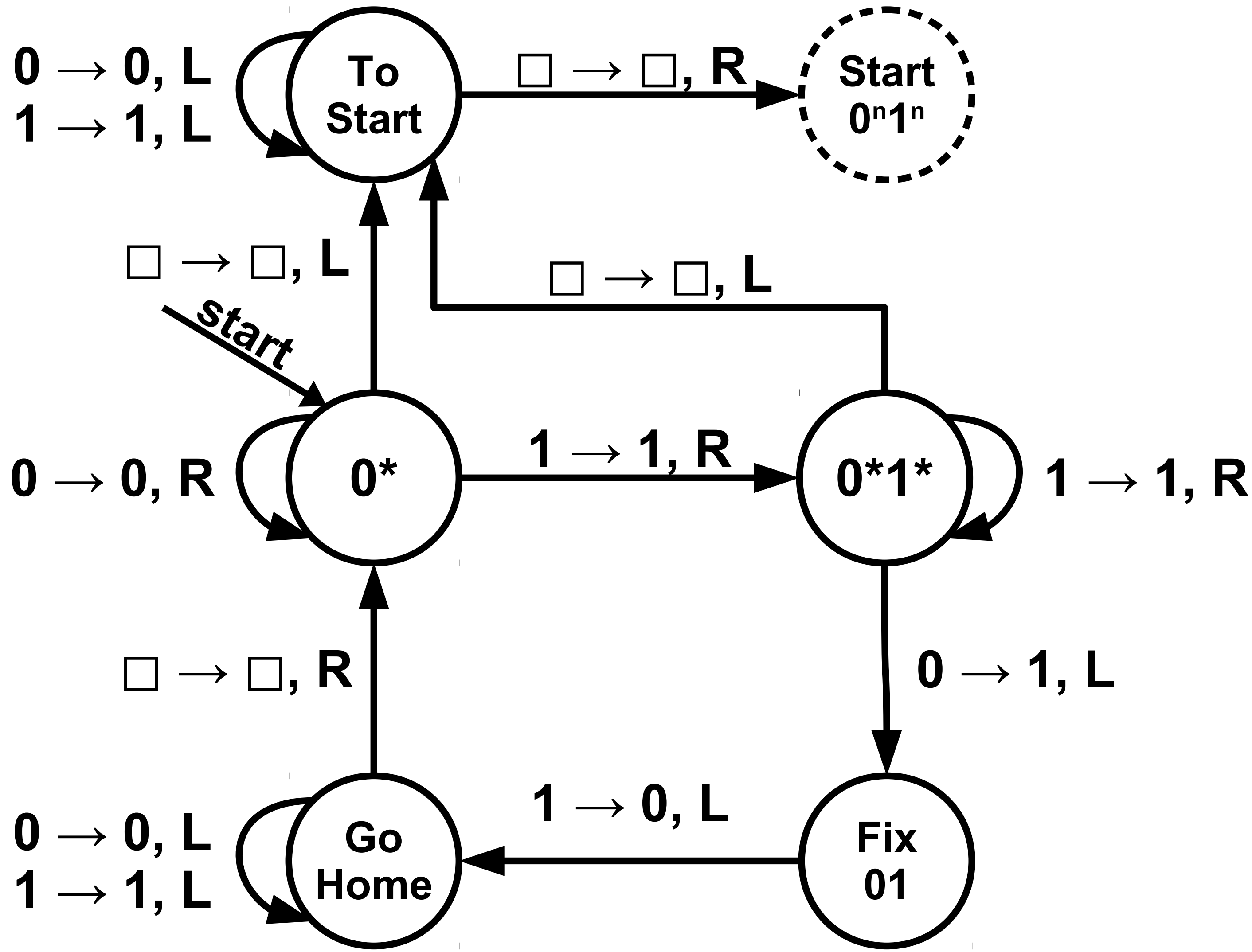
Observation 2: A string of 0s and 1s is ***not*** sorted if it contains 10 as a substring.

A Different Strategy



Idea: Repeatedly find a copy of 10 and replace it with 01.

Let's Build It!



0 → 0, L
1 → 1, L
To Start
□ → □, R
Start 0ⁿ1ⁿ
□ → □, L
start
□ → □, L
0 → 0, R
0*
1 → 1, R
0*1*
1 → 1, R
□ → □, R
0 → 1, L
1 → 0, L
0 → 0, L
1 → 1, L
Go Home
Fix 01

TM Subroutines

- A ***TM subroutine*** is a Turing machine that, instead of accepting or rejecting an input, does some sort of processing job.
- TM subroutines let us compose larger TMs out of smaller TMs, just as you'd write a larger program using lots of smaller helper functions.
- Here, we saw a TM subroutine that sorts a sequence of 0s and 1s into ascending order.

TM Subroutines

- Typically, when a subroutine is done running, you have it enter a state marked “done” with a dashed line around it.
- When we're composing multiple subroutines together – which we'll do in a bit – the idea is that we'll snap in some real state for the “done” state.

What other subroutines can we make?

TM Arithmetic

- Let's design a TM that, given a tape that looks like this:

...				1	3	7		4	2			...

ends up having the tape look like this:

...				1	7	9		0	0			...

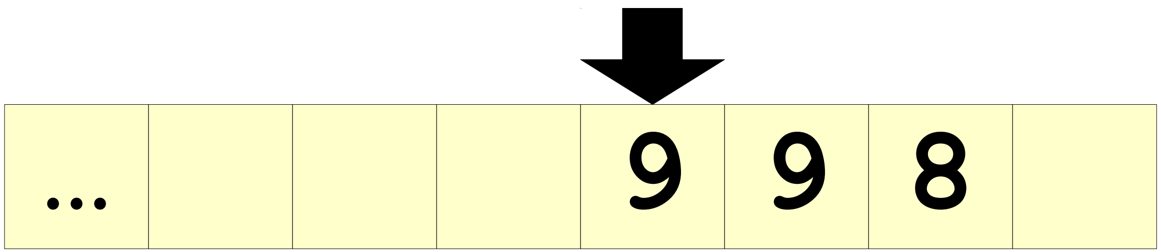
- In other words, we want to build a TM that can add two numbers.

TM Arithmetic

- There are many ways we could in principle design this TM.
- We're going to take the following approach:
  - First, we'll build a TM that increments a number.
  - Next, we'll build a TM that decrements a number.
  - Then, we'll combine them together, repeatedly decrementing the second number and adding one to the first number.

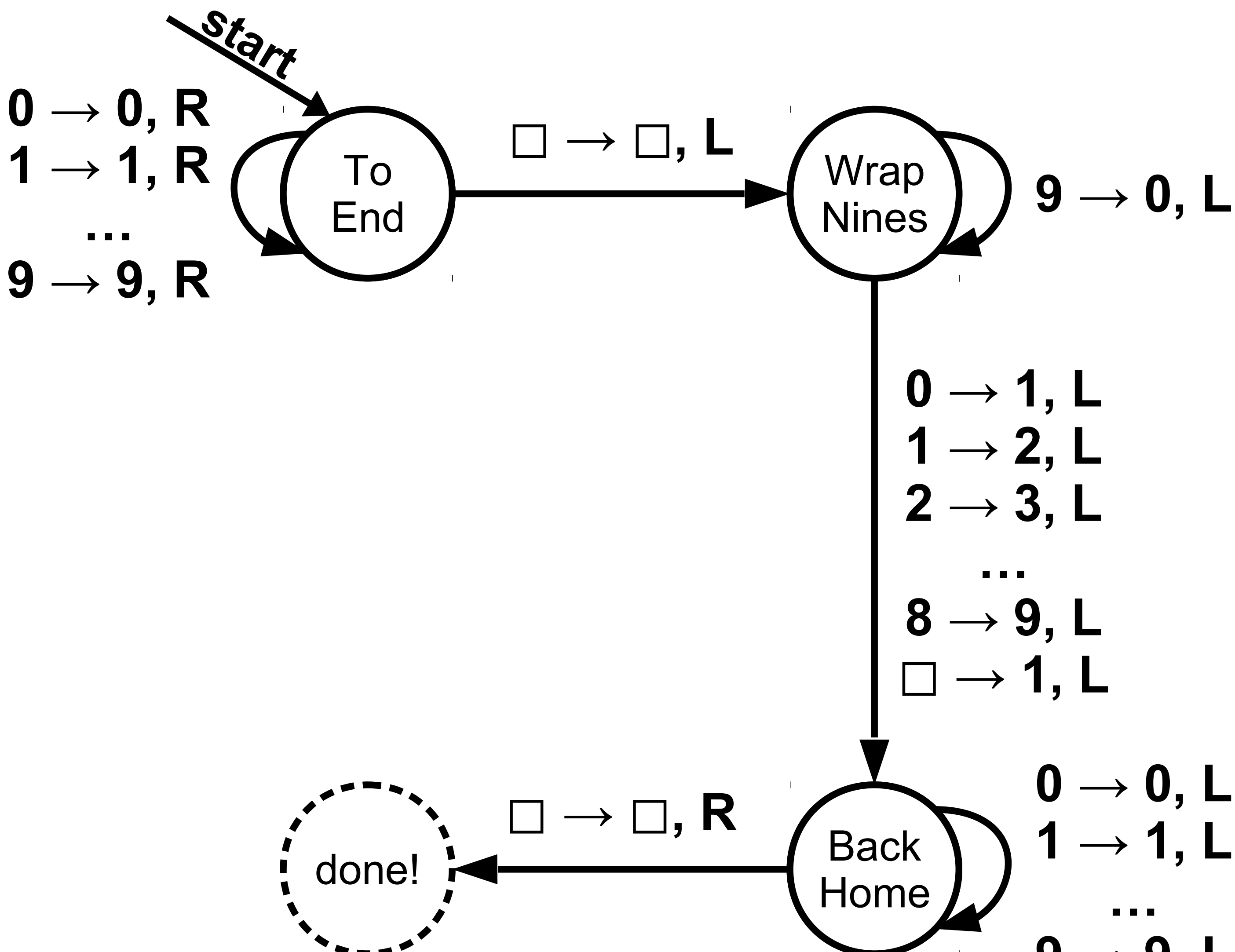
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is are at least two blanks to the left of the number, and
  - that there's at least one blank at the start of the number.
- The tape head will end at the start of the number after incrementing it.



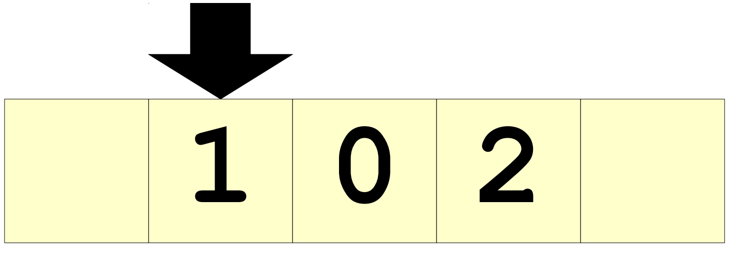
Incrementing Numbers

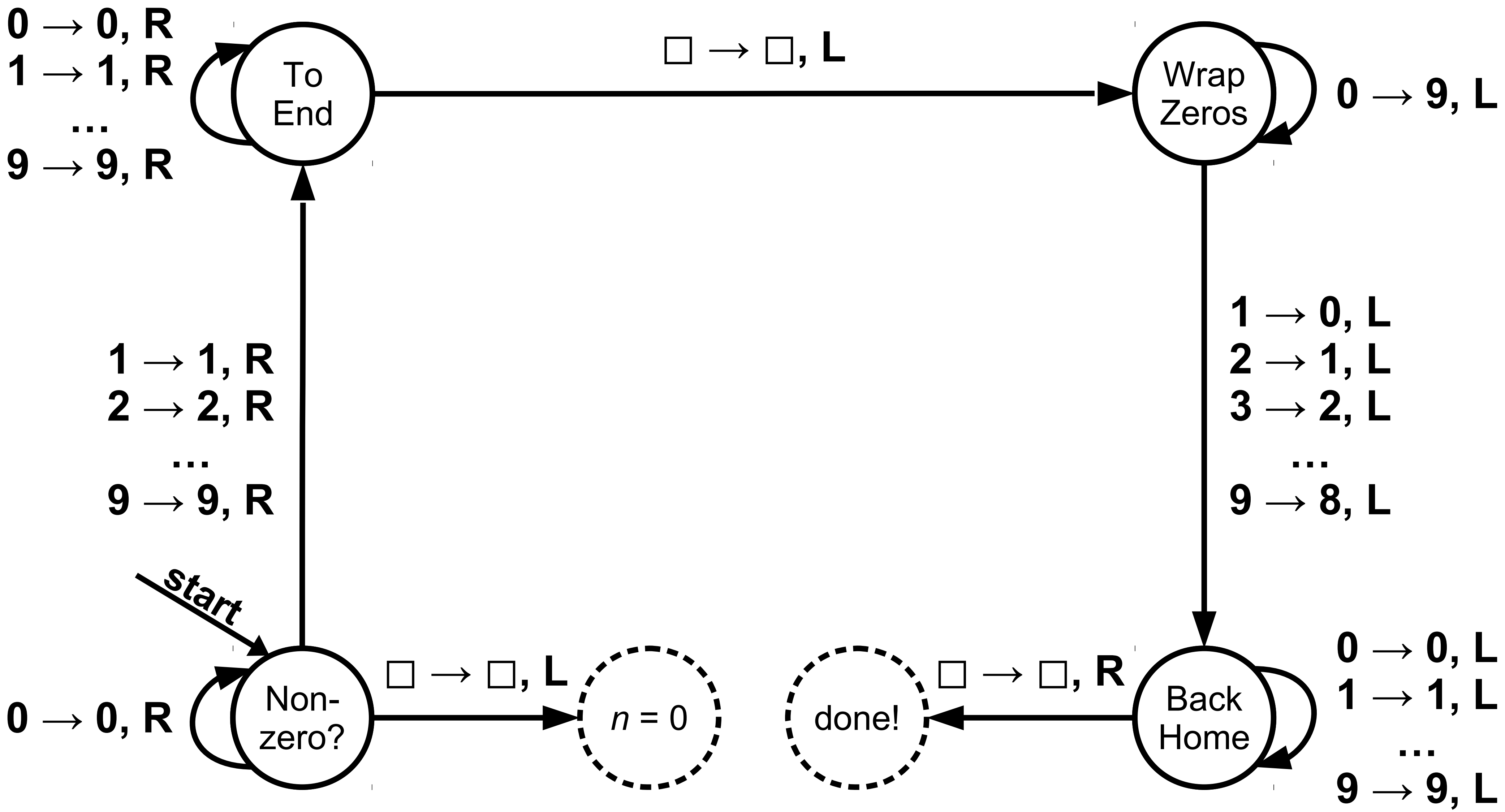
```
go to the end of the number;
while (the current digit is 9) {
    set the current digit to 0;
    back up one digit;
}
increment the current digit;
go to the start of the number;
```


start
0 → 0, R
1 → 1, R
…
9 → 9, R
To End
□ → □, L
Wrap Nines
9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
…
8 → 9, L
□ → 1, L
Back Home
0 → 0, L
1 → 1, L
…
9 → 9, L
□ → □, R
done!

Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.



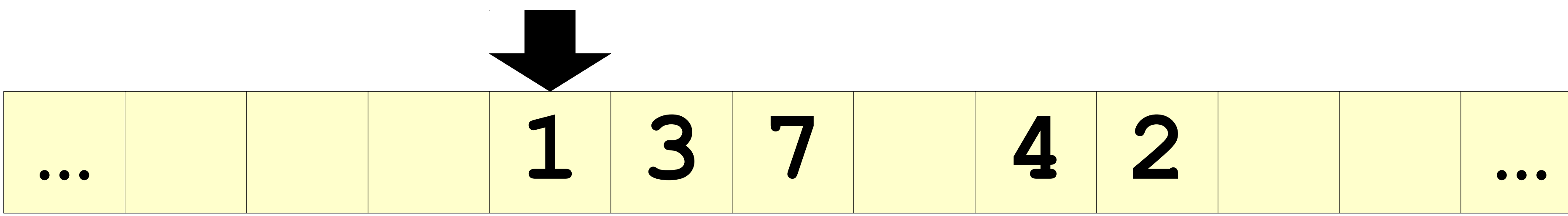

0 → 0, R
1 → 1, R
…
9 → 9, R
To End
□ → □, L
Wrap Zeros
0 → 9, L
1 → 1, R
2 → 2, R
…
9 → 9, R
1 → 0, L
2 → 1, L
3 → 2, L
…
9 → 8, L
start
0 → 0, R
Non-zero?
□ → □, L
n = 0
done!
□ → □, R
Back Home
0 → 0, L
1 → 1, L
…
9 → 9, L

TM Subroutines

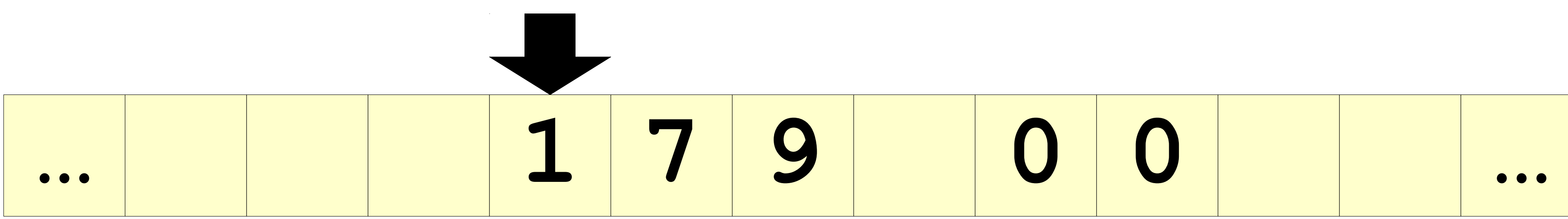
- Sometimes, a subroutine needs to report back some information about what happened.
- Just as a function can return multiple different values, we'll allow subroutines to have different "done" states.
- Each state can then be wired to a different state, so a TM using the subroutine can control what happens next.

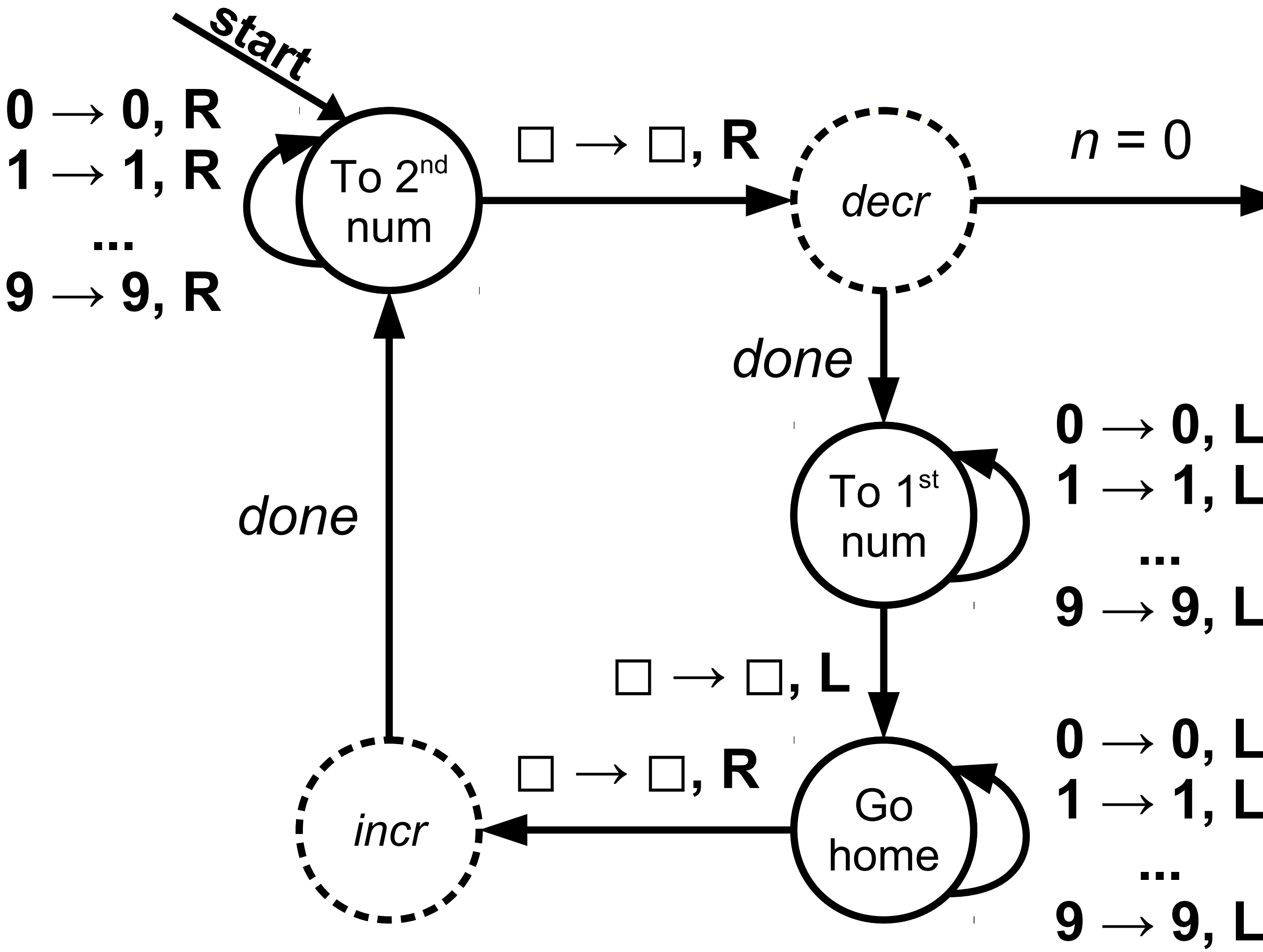
Putting it All Together

- Our goal is to build a TM that, given two numbers, adds those numbers together.
- Before:



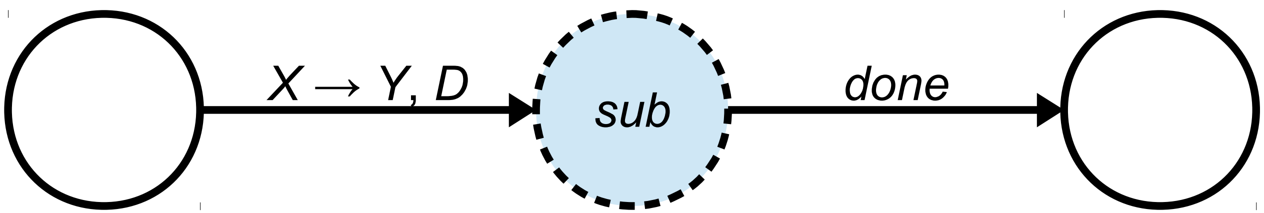
- After:




start
0 → 0, R
1 → 1, R
...
9 → 9, R
To 2nd num
□ → □, R
decr
n = 0
done!
done
To 1st num
0 → 0, L
1 → 1, L
...
9 → 9, L
□ → □, L
Go home
0 → 0, L
1 → 1, L
...
9 → 9, L
□ → □, R
incr
done

Using Subroutines

- Once you've built a subroutine, you can wire it into another TM with something that, schematically, looks like this:



- Intuitively, this corresponds to transitioning to the start state of the subroutine, then replacing the “done” state of the subroutine with the state at the end of the transition.

Time-Out for Announcements!

Problem Sets

- Problem Set Six was due at the start of class today using late days. Solutions are now available.
  - As always, we strongly recommend reading over the solution set – there's a lot of good advice in there!
- Problem Set Seven is due on Friday at the start of class.

Midterm Exam Logistics

- The second midterm exam is tomorrow, ***Tuesday, May 23rd***, from ***7:00PM – 10:00PM***. Locations are divvied up by last (family) name:
  - `Abb – Pag`: Go to Hewlett 200.
  - `Par – Tak`: Go to Sapp 114.
  - `Tan – Val`: Go to Hewlett 101.
  - `Var – Yim`: Go to Hewlett 102.
  - `You – Zuc`: Go to Hewlett 103.
- You're responsible for Lectures 00 – 13 and topics covered in PS1 – PS5. Later lectures and problem sets won't be tested. The focus is on PS3 – PS5 and Lectures 06 – 13.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

This is the part where I say nice things about you!

Your Questions

“Tech evolves really fast. What can I do to stay relevant in tech throughout my career ie not get replaced by young college grads in 10 years?”

In any skilled job – whether it's tech, medicine, law, etc. – you'll always be learning new techniques and staying on top of the latest developments. It's either institutionalized (e.g. law, medicine) or something that you'll pick up on the job (e.g. tech).

There's a lot of way to have fun while doing this. Attend conferences on topics you're interested in. Join a paper-reading group. Volunteer to work on projects in languages and frameworks you don't understand. Read blogs. Go on Stack Overflow. Or do some combination of these things!

“In your impression, how does Stanford's CS department feel about Silicon Valley and its controversies? I'm assuming they're mostly very pro-SV; is that right?”

There are very close ties between Silicon Valley and the entire School of Engineering – it's one of the reasons why the tech industry is so strong here and why the research program is so popular.

In my conversations with the faculty members here, I've found that people are generally pretty reasonable and don't blindly think that the tech industry is somehow perfect or that it can't do any wrong. It's typically more nuanced – most people generally like the idea of having a healthy tech sector, many folks wish that the focus was less on consumer tech, lots are upset about lack of diversity and toxic culture, etc. That happens alongside many of them working for tech companies or getting research funds from them.

“Beyond finite automata, and how discrete math applies to CS, what do you consider to be the relevance of proofs and mathematical reasoning in the real world?”

The discrete structures portion of this course is extremely valuable for thinking about how to model complex structures in the real world. Jure Leskovec's research program largely involves trying to model human behavior from a graph-theoretic perspective and using that to design interventions or otherwise predict user behavior. Strict orders come up all the time in searching and sorting algorithms, and equivalence relations are used in modeling hash tables.

Where we're going with computability theory – the limits of computation – is exceptionally valuable for understanding where you need to back up and search for alternate solutions. As you'll see, a lot of problems we'd love to be able to solve in practice are provably impossible, and when that happens it indicates that you need to switch directions or otherwise relax your constraints.

The sort of theory from CS109 and CS161 powers modern machine learning techniques and is responsible for things like cell phones (FFT), Google Maps (search algorithms) and the Internet (spanning trees). I'll talk more about this on our last day of class.

Back to CS103!

Main Question for Today:

Just how powerful are Turing machines?

How Powerful are TMs?

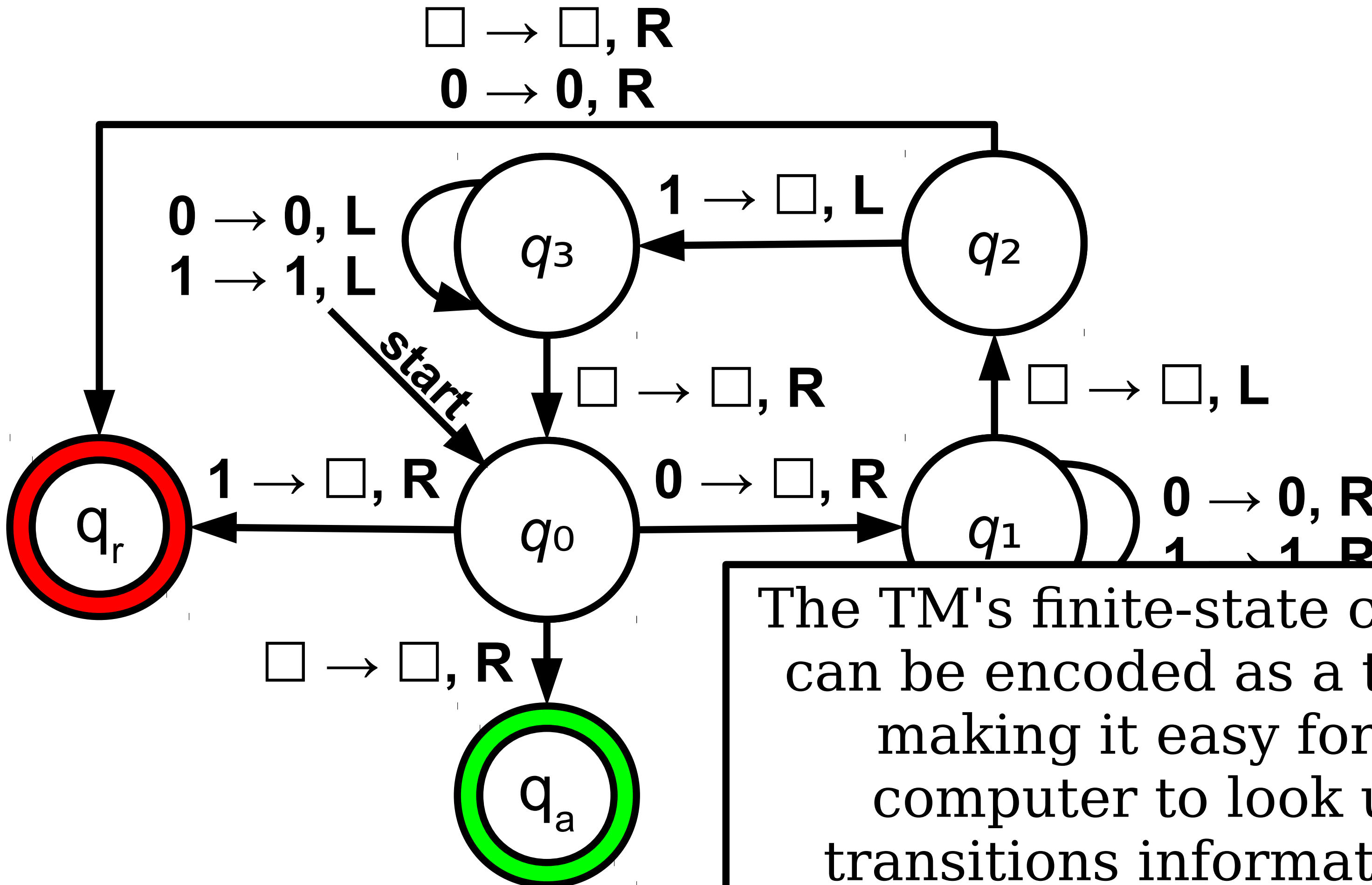
- Regular languages, intuitively, are as powerful as computers with finite memory.
- TMs by themselves seem like they can do a fair number of tasks, but it's unclear specifically what they can do.
- Let's explore their expressive power.

Real and "Ideal" Computers

- A real computer has memory limitations: you have a finite amount of RAM, a finite amount of disk space, etc.
- However, as computers get more and more powerful, the amount of memory available keeps increasing.
- An ***idealized computer*** is like a regular computer, but with unlimited RAM and disk space. It functions just like a regular computer, but never runs out of memory.

Claim 1: Idealized computers can simulate Turing machines.

"Anything that can be done with a TM can also be done with an unbounded-memory computer."



The TM's finite-state control can be encoded as a table, making it easy for a computer to look up transitions information.

	0			**1**			**□**		
q_0	q_1	□	**R**	q_r	□	**R**	q_a	□	**R**
q_1	q_1	**0**	**R**	q_1	**1**	**R**	q_2	□	**L**
q_2	q_r	**0**	**R**	q_3	□	**L**	q_r	□	**R**
q_3	q_3	**0**	**L**	q_3	**1**	**L**	q_0	□	**R**

Simulating a TM

- To simulate a TM, the computer would need to be able to keep track of
  - the finite-state control,
  - the current state,
  - the position of the tape head, and
  - the tape contents.
- The tape contents are infinite, but that's because there are infinitely many blanks on both sides.
- We only need to store the “interesting” part of the tape (the parts that have been read from or written to so far.)

1	7	9		0	0

Claim 2: Turing machines can simulate idealized computers.

“Anything that can be done with an unbounded-memory computer can be done with a TM.”

What We've Seen

- TMs can
  - implement loops (basically, every TM we've seen).
  - make function calls (subroutines).
  - keep track of natural numbers (written in unary or in decimal on the tape).
  - perform elementary arithmetic (equality testing, multiplication, addition, increment, decrement, etc.).
  - perform if/else tests (different transitions based on different cases).

What Else Can TMs Do?

- Maintain variables.
  - Have a dedicated part of the tape where the variables are stored.
  - We've seen this before: take a look at our machine for composite numbers, or for increment/decrement.
- Maintain arrays and linked structures.
  - Divide the tape into different regions corresponding to memory locations.
  - Represent arrays and linked structures by keeping track of the ID of one of those regions.

A CS107 Perspective

- Internally, computers execute by using basic operations like
  - simple arithmetic,
  - memory reads and writes,
  - branches and jumps,
  - register operations,
  - etc.
- Each of these are simple enough that they could be simulated by a Turing machine.

A Leap of Faith

- It may require a leap of faith, but anything you can do a computer (excluding randomness and user input) can be performed by a Turing machine.
- The resulting TM might be colossal, or really slow, or both, but it would still faithfully simulate the computer.
- We're going to take this as an article of faith in CS103. If you curious for more details, come talk to me after class.

Just how powerful ***are*** Turing machines?

Effective Computation

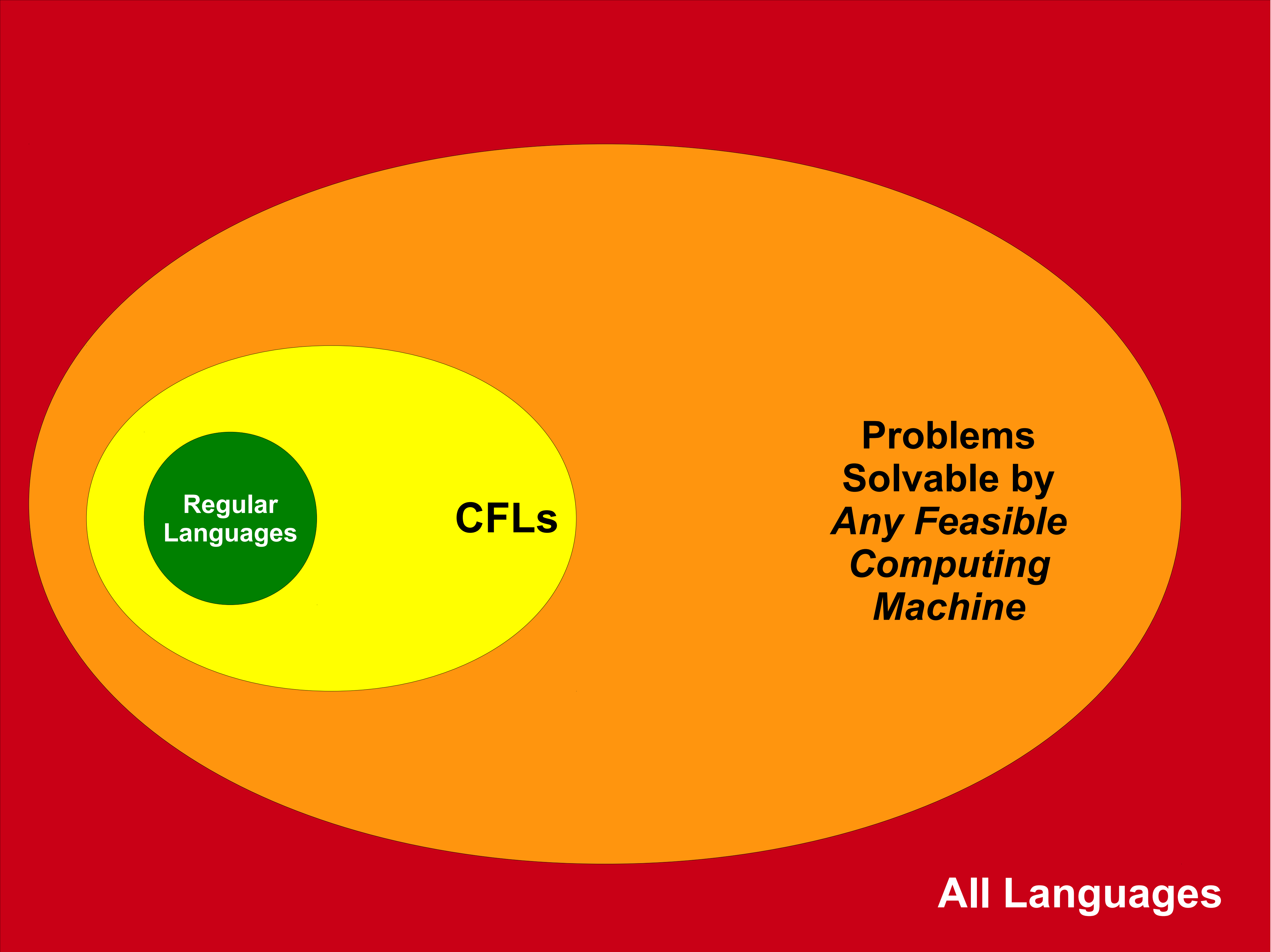
- An ***effective method of computation*** is a form of computation with the following properties:
  - The computation consists of a set of steps.
  - There are fixed rules governing how one step leads to the next.
  - Any computation that yields an answer does so in finitely many steps.
  - Any computation that yields an answer always yields the correct answer.
- This is not a formal definition. Rather, it's a set of properties we expect out of a computational system.

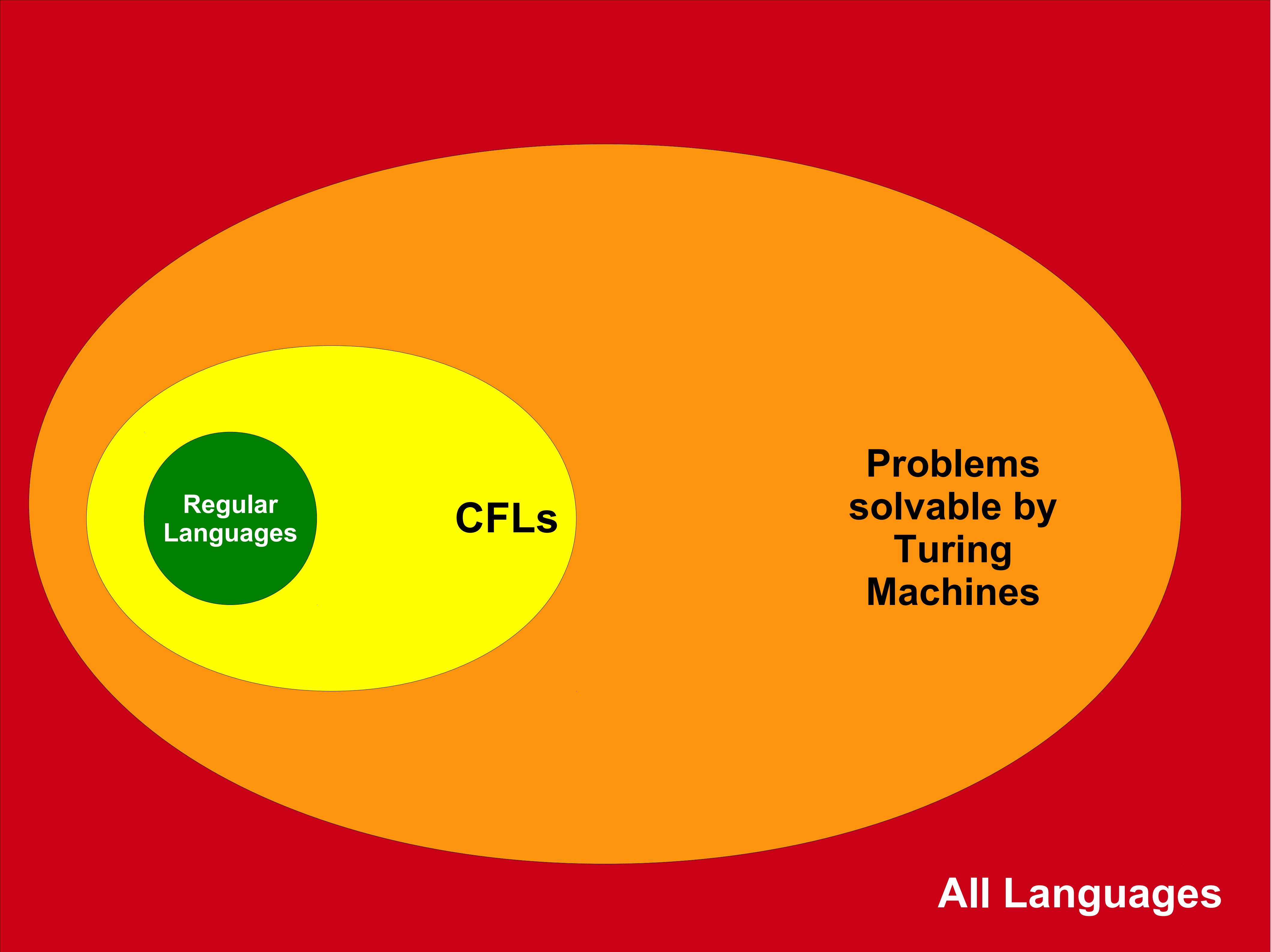
The ***Church-Turing Thesis*** claims that

every effective method of computation is either equivalent to or weaker than a Turing machine.

"This is not a theorem – it is a falsifiable scientific hypothesis. And it has been thoroughly tested!"

- Ryan Williams


Regular Languages
CFLs
Problems Solvable by *Any Feasible Computing Machine*
All Languages


Regular Languages
CFLs
Problems solvable by Turing Machines
All Languages

TMs ≈ Computers

- Because Turing machines have the same computational powers as regular computers, we can (essentially) reason about Turing machines by reasoning about actual computer programs.
- Going forward, we're going to switch back and forth between TMs and computer programs based on whatever is most appropriate.
- In fact, our eventual proofs about the existence of impossible problems will involve a good amount of pseudocode. Stay tuned for details!