Week 5 Tutorial

Cardinality and Graph Theory
Part 1: *Graph Theory Warmup*
Consider the following FOL statements about some graph $G = (V, E)$:

a) $\forall u \in V. \forall v \in V. \{u, v\} \notin E$

b) $\exists u \in V. \forall v \in V. (u \neq v \rightarrow \{u, v\} \in E)$

c) $\forall u \in V. \exists v \in V. (\{u, v\} \in E \land (\forall w \in V. w \neq v \rightarrow \{u, w\} \notin E))$

1. Translate these FOL statements into English. Please use graph theory terms (adjacent, connected, degree, etc.) where appropriate.

*Fill in answer on Gradescope!*
Part 2: *Graph Theory*
From the problem set:

An undirected graph $G = (V, E)$ is called **bipartite** if there exist two sets $V_1$ and $V_2$ such that

- every node $v \in V$ belongs to exactly one of $V_1$ and $V_2$, and
- every edge $e \in E$ has one endpoint in $V_1$ and the other in $V_2$. 
From the problem set:

An undirected graph $G = (V, E)$ is called **bipartite** if there exist two sets $V_1$ and $V_2$ such that

- every node $v \in V$ belongs to exactly one of $V_1$ and $V_2$, and
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Now, a new definition:

Let $d(u, v)$ denote the length of the shortest path in $G$ from $u$ to $v$. 
From the problem set:

An undirected graph \( G = (V, E) \) is called \textit{bipartite} if there exist two sets \( V_1 \) and \( V_2 \) such that

- every node \( v \in V \) belongs to exactly one of \( V_1 \) and \( V_2 \), and
- every edge \( e \in E \) has one endpoint in \( V_1 \) and the other in \( V_2 \).

Now, a new definition:

Let \( d(u, v) \) denote the length of the shortest path in \( G \) from \( u \) to \( v \).

\textbf{Theorem:} A connected graph \( G \) is bipartite if and only if for every vertex \( v \), there is no edge \( \{u, w\} \) where \( d(v, u) = d(v, w) \).
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

We'll now prove this direction of the biconditional.
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

Choose an arbitrary connected graph $G = (V, E)$ where for every vertex $v$, $d(v, u) \neq d(v, w)$ for any choice of $\{u, w\} \in E$. Pick a node $v \in V$ to use as an “anchor point” and define the following two sets based on $v$:

- $V_1 = \{ x \in V \mid d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \}$
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

Choose an arbitrary connected graph $G = (V, E)$ where for every vertex $v$, $d(v, u) \neq d(v, w)$ for any choice of $\{u, w\} \in E$. Pick a node $v \in V$ to use as an “anchor point” and define the following two sets based on $v$:

- $V_1 = \{ x \in V | d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V | d(x, v) \text{ is odd} \}$

2a) Refer back to the definition of a bipartite graph and list out the **three** things you need to prove in order to prove that $G$ is bipartite.

Hint: To prove “exactly one” there are two things you need to show. What are they?

*Fill in answer on Gradescope!*
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

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- $V_1 = \{ x \in V \mid d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \}$

**Want to Show:**

1. Every node is in *at least* one of $V_1$ and $V_2$
2. Every node is in *at most* one of $V_1$ and $V_2$
3. Every edge has one endpoint in $V_1$ and one in $V_2$
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

Choose an arbitrary connected graph $G = (V, E)$ where for every vertex $v$, $d(v, u) \neq d(v, w)$ for any choice of $\{u, w\} \in E$. Pick a node $v \in V$ to use as an “anchor point” and define the following two sets based on $v$:

- $V_1 = \{ x \in V \mid d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \}$

**Want to Show:**

1. Every node is in *at least* one of $V_1$ and $V_2$

2b) Explain why this assertion holds for the choices of $V_1$ and $V_2$ given above.

*Fill in answer on Gradescope!*
Want to Show:

1. Every node is in \textit{at least} one of $V_1$ and $V_2$

Pick an arbitrary node $x$
**Want to Show:**

1. Every node is in *at least* one of $V_1$ and $V_2$

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Pick an arbitrary node $x$.

Since $\mathcal{G}$ is connected, we know that there's at least one path from $x$ to our anchor point $v$.

(This dotted line just represents that there's some series of edges you can follow to get from $x$ to $v$)
Want to Show:

1. Every node is in \textit{at least} one of $V_1$ and $V_2$

Pick an arbitrary node $x$

Since $G$ is connected, we know that there's at least one path from $x$ to our anchor point $v$.

- $V_1 = \{ x \in V \mid d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \}$

If $d(x, v)$ is even, then $x$ is in $V_1$.
Otherwise, $x$ is in $V_2$. 
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

Pick any node $v \in V$ to use as an “anchor point” and define the following two sets based on $v$:

- \( V_1 = \{ x \in V \mid d(x, v) \text{ is even} \} \)
- \( V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \} \)

**Want to Show:**

2. Every node is in *at most* one of $V_1$ and $V_2$

2c) Explain why this assertion holds for the choices of $V_1$ and $V_2$ given above.

*Fill in answer on Gradescope!*
Want to Show:

2. Every node is in \textit{at most} one of $V_1$ and $V_2$

Pick an arbitrary node $x$
Want to Show:

2. Every node is in \textit{at most} one of $V_1$ and $V_2$

- $V_1 = \{ x \in V \mid d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \}$
**Want to Show:**

2. Every node is in *at most* one of \( V_1 \) and \( V_2 \)

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**Pick an arbitrary node** \( x \)  

Let's suppose that \( x \) is in both \( V_1 \) and \( V_2 \).

\[
d(x, v) \quad \cdots \quad v
\]

- \( V_1 = \{ x \in V \mid d(x, v) \text{ is even} \} \)
- \( V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \} \)

This would mean that \( d(x, v) \) is both even and odd, which is impossible.
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

Pick any node $v \in V$ to use as an “anchor point” and define the following two sets based on $v$:

- $V_1 = \{ x \in V \mid d(x, v) \text{ is even} \}$
- $V_2 = \{ x \in V \mid d(x, v) \text{ is odd} \}$

**Want to Show:**

3. Every edge has one endpoint in $V_1$ and one in $V_2$

2d) Explain why this assertion holds for the choices of $V_1$ and $V_2$ given above. Click to the next few slides for some hints.

*Fill in answer on Gradescope!*
Want to Show:
3. Every edge has one endpoint in $V_1$ and one in $V_2$

Hint 1:

Pick an arbitrary edge $\{u, w\}$ and think about $d(v, u)$, $d(v, w)$, and $d(u, w)$
**Want to Show:**

3. Every edge has one endpoint in $V_1$ and one in $V_2$

**Hint 2:**

$|d(v, u) - d(v, w)| \leq 1$

Why is this the case? Suppose for the sake of contradiction that $|d(v, u) - d(v, w)| > 1$

These vertical bars mean absolute value :)
**Want to Show:**

3. Every edge has one endpoint in $V_1$ and one in $V_2$

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**Hint 3:**

$$|d(v, u) - d(v, w)| \leq 1$$

We’ve assumed that $d(v, u) \neq d(v, w)$, so the difference $|d(v, u) - d(v, w)|$ can’t be 0. What does that mean about the statement above?
**Want to Show:**

3. Every edge has one endpoint in $V_1$ and one in $V_2$

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**Hint 4:**

$$d(v, u) \quad \ldots \quad u \quad \ldots \quad d(u, w)$$

$$d(v, w) \quad \ldots \quad v \quad \ldots \quad d(v, w)$$

$$|d(v, u) - d(v, w)| = 1$$

Think about what this tells you about whether or not $d(v, u)$ and $d(v, w)$ can both be even or odd.
**Theorem:** If $G$ is a connected graph where for every vertex $v$, there is no edge $\{u, w\}$ where $d(v, u) = d(v, w)$, then $G$ is bipartite.

**Proof:** To prove that $G$ is bipartite, will prove that every node in $V$ belongs to exactly one of $V_1$ and $V_2$ and that every edge in $E$ has one endpoint in $V_1$ and one in $V_2$.

We’ll start by showing that every node in $V$ belongs to exactly one of $V_1$ and $V_2$. Pick an arbitrary node $x \in V$. To see that $x$ belongs to at least one of $V_1$ and $V_2$, notice that $G$ is connected so there is at least one path from $x$ to $v$. If $d(v, x)$ is even, then $x$ is in $V_1$, otherwise $d(v, x)$ is odd and $x$ is in $V_2$. Additionally, $x$ can be in at most one of $V_1$ and $V_2$. To see why, suppose for the sake of contradiction that $x \in V_1$ and $x \in V_2$. This would mean that $d(v, x)$ is both even and odd, which is impossible.

(continued on next slide ... )
Now we will show that the edges of $G$ run between $V_1$ and $V_2$.

Pick an arbitrary edge $\{u, w\} \in E$.

We’ll first show that $d(v, u)$ and $d(v, w)$ differ by no more than 1. Suppose for the sake of contradiction that $|d(v, u) - d(v, w)| > 1$, and without loss of generality let $d(v, u) > d(v, w)$. This would mean that $d(v, u) \geq d(v, w) + 2$. However, we can find a shorter path from $v$ to $u$ by following a path of length $d(v, w)$ from $v$ to $w$ and then following the edge $\{u, w\}$ for a total length of $d(v, w) + 1$, contradicting our assertion that $d(v, u) \geq d(v, w) + 2$.

Thus, we know that $|d(v, u) - d(v, w)| \leq 1$. Furthermore, since we assumed that $d(v, u) \neq d(v, w)$, so the difference $|d(v, u) - d(v, w)|$ can’t be 0 and we must have

$$|d(v, u) - d(v, w)| = 1.$$ 

Since this difference is odd, we know that that the integers $d(v, u)$ and $d(v, w)$ have opposite parity, allowing us to conclude that $u$ and $w$ cannot both be in $V_1$ or in $V_2$. Therefore we’ve shown that $G$ must be bipartite, as required. ■