Problems for Week Four

Problem One: Concept Checks

You know the drill. Here’s a review from the topics from last week.

i. Give two examples of binary relations over the set $\mathbb{N}$.

ii. What three properties must a binary relation have to have in order to be an equivalence relation? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation $R$ has that property.

iii. If $R$ is an equivalence relation over a set $A$ and $a$ is an element of $A$, what does the notation $[a]_R$ mean? Intuitively, what does it represent?

iv. What three properties must a binary relation have to have in order to be a strict order? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation $R$ has that property.

v. What is a Hasse diagram? Give an example.

vi. What does the notation $f : A \rightarrow B$ mean?

vii. Let $f : A \rightarrow B$ be a function. Express, in first-order logic, what property $f$ has to satisfy to be an injection. Then, based on the structure of that formula, explain how you would write a proof that $f$ is injective.

viii. Negate your statement from part (vii) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that $f$ is not injective.

ix. Let $f : A \rightarrow B$ be a function. Express, in first-order logic, what property $f$ has to satisfy to be a surjection. Then, based on the structure of that formula, explain how you would write a proof that $f$ is surjective.

x. Negate your statement from part (ix) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that $f$ is not surjective.

xi. Let $f : A \rightarrow B$ be a function. What properties must $f$ have to be a bijection? How would you write a proof that $f$ is bijective?

xii. What would you need to prove to show that $f$ is not a bijection?
Problem Two: Equivalence Relations

This question explores various properties of equivalence relations.

i. In lecture, we proved that the binary relation \( \sim \) over \( \mathbb{Z} \) defined as follows is an equivalence relation:

\[
a \sim b \quad \text{if} \quad a + b \text{ is even.}
\]

Consider this new relation \( \# \) defined over \( \mathbb{Z} \):

\[
a \# b \quad \text{if} \quad a + b \text{ is odd.}
\]

Is \( \# \) an equivalence relation? If so, prove it. If not, disprove it.

ii. How many equivalence classes are there for the \( \sim \) relation defined above? What are they?

Problem Three: Inverse Relations

Let \( R \) be a binary relation over a set \( A \). We can define a new relation over \( A \) called the inverse relation of \( R \), denoted \( R^{-1} \), as follows:

\[
xR^{-1}y \quad \text{if} \quad yRx
\]

This question explores properties of inverse relations.

i. What is the inverse of the \( < \) relation over \( \mathbb{Z} \)? Briefly justify your answer.

ii. What is the inverse of the \( = \) relation over \( \mathbb{Z} \)? Briefly justify your answer.

iii. Prove or disprove: if \( R \) is an equivalence relation over \( A \), then \( R^{-1} \) is an equivalence relation over \( A \).

Problem Four: Monotone Functions

A function \( f : \mathbb{R} \to \mathbb{R} \) is called monotone increasing if the following is true:

\[
\forall x \in \mathbb{R}. \forall y \in \mathbb{R}. (x < y \to f(x) < f(y))
\]

This problem explores properties of monotone increasing functions.

i. Prove or disprove: every monotone increasing function is injective.

ii. Prove or disprove: every injective function from \( \mathbb{R} \) to \( \mathbb{R} \) is monotone increasing.

Problem Five: Involutions

A function \( f : A \to A \) is called an involution if \( f(f(x)) = x \) for all \( x \in A \).

i. Find three different examples of involutions from \( \mathbb{Z} \) to \( \mathbb{Z} \). Briefly justify your answers.

ii. Prove that if \( f \) is an involution, then \( f \) is a bijection.

Problem Six: Functions and Relations – Together!

Let \( f : A \to B \) be an arbitrary function. Define a new binary relation \( \sim \) over \( A \) as follows:

\[
x \sim y \quad \text{if} \quad f(x) = f(y)
\]

Prove that \( \sim \) is an equivalence relation over \( A \).