Induction
What we’ve covered:

- Induction
- Variations on Induction
  - Multiple base cases
  - Larger step sizes
- Complete Induction
The big question in a proof by induction: *How can I leverage a smaller result to help me prove a bigger result?*
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How do we do this?

Is the statement we’re trying to prove universally or existentially quantified? Based on that, what should we do here?

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In this example, we’re trying to prove an existentially quantified statement.

The easiest way to do so is to just find an object of larger size with the desired property.

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How are we going to do that? What tools do we have at hand?

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We get to start by assuming this existential statement, which means that we assume there exists a smaller object that has our desired property.

And our goal is to show that $P$ is true for a bigger version of the problem.

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Given this assumption that we have a concrete object to work with, it makes sense that we can try to take that smaller object and somehow grow it into a larger object.

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Existentially Quantified $P(n)$
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1. Start with a smaller object
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2. Apply inductive hypothesis

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3. Use it to construct a larger object
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1. Start with a smaller object
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We refer to this technique as “Build Up”
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How do we do this? Is the statement we’re trying to prove universally or existentially quantified? Based on that, what should we do here?

$P(k+1) = “$any tree with $k+1$ nodes has $k$ edges.$“$
In this example, we're trying to prove a universally quantified statement.

Thus, we should pick an arbitrary object of larger size and show why it has our desired property.

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We can’t directly apply our inductive hypothesis to our larger object, because our assumption only applies to smaller objects.

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Universally Quantified $P(n)$
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We refer to this technique as "Build Down"
How Not to Induct
How Not to Induct

1. Start with a smaller object
2. Apply inductive hypothesis
3. Use it to construct a larger object

Why can’t we “Build Up” with universally quantified statements?
How Not to Induct

Remember that our end goal is to prove a universal statement about all objects of size $k+1$. It would thus be incorrect to start by picking an arbitrary object of size $k$ and modifying it because you’d have to explain why your conclusion actually applies to all objects of size $k+1$.

1. Start with a smaller object
2. Apply inductive hypothesis
3. Use it to construct a larger object
Build Up vs. Build Down

- Build up if $P(n)$ is existentially qualified
  - We can use our inductive hypothesis (there exists an object of size $k$) to prove that there exists an object of size $k+1$
- Build down if $P(n)$ is universally qualified
  - Our inductive hypothesis (for all objects of size $k$, some property is true) doesn’t apply to an object of size $k+1$
Let’s play a game!
Rules

- Start with a pile of $n$ coins for some $n \geq 0$
- Players take turns removing between 1 and 5 coins from the pile
- The player who has no more coins to remove loses the game

Interestingly, if the pile begins with a multiple of 6 coins in it, the second player can always win if they play correctly.
Rules

- Start with a pile of $n$ coins for some $n \geq 0$
- Players take turns removing between 1 and 5 coins from the pile
- The player who has no more coins to remove loses the game

- Interestingly, if the pile begins with a multiple of 6 coins in it, the second player can always win if they play correctly - give it a try!
What’s the strategy?
Some Observations

- If it’s the first player’s turn and there are no coins left, then the second player wins.

Player 1

Player 2
Some Observations

- If it’s the first player’s turn and there are no coins left, then the second player wins.
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- If it’s the first player’s turn and there are no coins left, then the second player wins.
- If we start with 6 coins, player 1 has to remove some but not all of the coins.
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Some Observations

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Some Observations

- If it’s the first player’s turn and there are no coins left, then the second player wins.
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins.
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• If it’s the first player’s turn and there are no coins left, then the second player wins.

• If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.
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- If it’s the first player’s turn and there are no coins left, then the second player wins.
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.
- What happens when there are 12 coins?

![Diagram showing player 1 and player 2 with coins]
Some Observations

- If it’s the first player’s turn and there are no coins left, then the second player wins.
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.
- What happens when there are 12 coins? Player 1 removes some coins.
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• If it’s the first player’s turn and there are no coins left, then the second player wins.

• If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.

• What happens when there are 12 coins? Player 1 removes some coins, but then player 2 can remove the right number of coins to leave 6 remaining.
Some Observations

• If it’s the first player’s turn and there are no coins left, then the second player wins.

• If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.

• What happens when there are 12 coins? Player 1 removes some coins, but then player 2 can remove the right number of coins to leave 6 remaining. It’s player 1’s turn again and there are 6 coins, again a known winning state.
**Strategy**: The second player can win by making the total number of coins removed by their move and the first player’s move come out to 6.
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It is a great idea to try small cases before jumping into a formal proof. It will be much easier to formalize the logic here now that you have a feel for how to play the game.
For all games where the number of coins is a multiple of 6, the second player can always win if they play correctly.

What is P(n)?

What does the problem size “n” in P(n) represent?

What is the base case?

What is the step size?
For all games where the number of coins is a multiple of 6, the second player can always win if they play correctly.

What is \( P(n) \)?

*Let \( P(n) \) be the statement “if the game is played with the pile containing \( n \) coins, the second player can always win if she plays correctly.”*

What does the problem size “\( n \)” in \( P(n) \) represent?

*The problem size is the number of coins.*

What is the base case?

*The base case is \( n=0 \), the simplest possible case of the game is when you start with no coins.*

What is the step size?

*We want to show the result is true for multiples of 6, so we’ll take steps of size 6.*
$P(n) =$ “if the game is played with the pile containing $n$ coins, the second player can always win if they play correctly.”

Is $P(n)$ universally or existentially quantified? Based on that, should we build up or build down?

$P(k)$

$P(k+6)$
$P(n) = \text{“if the game is played with the pile containing } n \text{ coins, the second player can always win if they play correctly.”}$

Is $P(n)$ universally or existentially quantified? Based on that, should we build up or build down?

$P(n)$ is universally qualified so we should build down (start with a game of size $k+6$ and figure out how to reduce it to a game of size $k$).
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Notice how even if we had no idea how to accomplish this yet, we can still answer all of these questions and set up the proof correctly – this is huge!
P(n) = “if the game is played with the pile containing \( n \) coins, the second player can always win if they play correctly.”

Assume P(k)

(If the game is played with \( k \) coins, the second player can always win)

Prove P(k+6)

(If the game is played with \( k+6 \) coins, the second player can always win)
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Assume $P(k)$

(If the game is played with $k$ coins, the second player can always win)

We want to take a game with $k+6$ coins and explain the strategy for reducing that game into one with just $k$ stones so that we can apply the inductive hypothesis

Prove $P(k+6)$

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**Strategy:** The second player can win by making the total number of coins removed by their move and the first player’s move come out to 6.
P(n) = “if the game is played with the pile containing n coins, the second player can always win if they play correctly.”

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(If the game is played with k coins, the second player can always win)

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Prove $P(k+6)$

(If the game is played with $k+6$ coins, the second player can always win)

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Prove P(k+6)
(If the game is played with \( k+6 \) coins, the second player can always win)

Now there are \( k \) coins remaining, meaning that we can apply our inductive hypothesis
Setting Up a Proof by Induction

- **Identify a \( P(n) \)**
  - Make sure \( P(n) \) is a predicate, not a number.

- **Identify and prove a base case**
  - This is the simplest possible case for \( P(n) \)

- **Prove the inductive step**
  - What is the step size? Are you building up or building down? Do you need complete induction?