Problems for Week Three

Problem One: Concept Checks

As a warm-up, take a few minutes to answer these questions before moving on.

i. There are five propositional connectives besides ⊤ and ⊥. What are they? Write out their truth tables. If you can do this completely from memory, fantastic! Do that. If you have to look things up, no worries! Use the Truth Table Tool, but try to make an effort to learn these tables as soon as you can.

ii. What is the negation of the formula $p \rightarrow q$? Repeat this exercise for the four remaining propositional connectives.

iii. In a propositional logic formula, what does each variable represent? In a first-order logic formula, what does each variable represent?

iv. What is the difference between a predicate and a function?

v. Can predicates and functions be applied to objects?

vi. Can predicates and functions be applied to propositions?

vii. There's a propositional connective that often pairs with the ∀ quantifier. Which is it?

viii. There's a propositional connective that often pairs with the ∃ quantifier. Which is it?

ix. In first-order logic, is the equality symbol (=) a predicate, a function, both, or neither?

Problem Two: Implications are Weird

The "implies" connective → is one of the stranger connectives. Below are a series of statements regarding implications. For each statement, confirm that it is indeed true, then briefly explain why.

i. For any propositions P and Q, the following is always true: $(P \rightarrow Q) \vee (Q \rightarrow P)$.

ii. More generally, for any propositions P, Q, and R, the following statement is always true: $(P \rightarrow Q) \vee (Q \rightarrow R)$.

Problem Three: Designing Propositional Formulas

Below is a series of English descriptions of relations among propositional variables. For each description, write a propositional formula that precisely encodes that relation. Then, briefly explain the intuition behind your formula. Try to see if you can come up with the simplest formula possible. If you find yourself writing out something extremely long, chances are you're missing something easier. All of these problems have formulas that can be cleanly written on a single line.

    i.   For the variables p, q, and r: at least two of p, q, and r are true.

    ii. For the variables a, b, c, and d: If any of the variables are true, then all the variables that follow it alphabetically in the English alphabet are also true.

Problem Four: True or False?

Below is a list of statements written in first-order logic. For each statement, translate it into English, then decide whether it's true or false.

    i. $\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ n < m$

    ii. $\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ n < m$

    iii. $\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ (n < m \rightarrow \exists p \in \mathbb{N}.\ (n < p \wedge p < m))$

    iv. $\forall n \in \mathbb{R}.\ \forall m \in \mathbb{R}.\ (n < m \rightarrow \exists p \in \mathbb{R}.\ (n < p \wedge p < m))$

    v. $\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ \exists p \in \mathbb{N}.\ (n = p \cdot m)$

    vi. $\forall n \in \mathbb{R}.\ \forall m \in \mathbb{R}.\ \exists p \in \mathbb{R}.\ (n = p \cdot m)$

Problem Five: Interpersonal Dynamics

The diagram to the right represents a set of people named A, B, C, and D. If there's an arrow from a person x to a person y, then person x loves person y. We'll denote this by writing $Loves(x, y)$. Below is a list of formulas in first-order logic about the above picture. In those formulas, the letter P represents the set of all the people. For each formula, determine whether that formula is true or false.
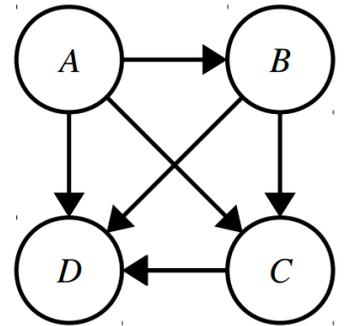
    i. $\forall x \in P.\ \forall y \in P.\ (Loves(x, y) \vee Loves(y, x))$

    ii. $\forall x \in P.\ \forall y \in P.\ (x \neq y \rightarrow Loves(x, y) \vee Loves(y, x))$

    iii. $\forall x \in P.\ \forall y \in P.\ (x \neq y \rightarrow (Loves(x, y) \leftrightarrow \neg Loves(y, x)))$

    iv. $\exists x \in P.\ \forall y \in P.\ (Loves(x, y))$

    v. $\exists x \in P.\ \forall y \in P.\ (x \neq y \rightarrow Loves(x, y))$

    vi. $\forall y \in P.\ \exists x \in P.\ (Loves(x, y))$

    vii. $\forall y \in P.\ \exists x \in P.\ (x \neq y \wedge Loves(x, y))$

    viii. $\exists x \in P.\ \forall y \in P.\ (\neg Loves(x, y))$

Problem Six: Negating Statements

For each of the first-order logic formulas below, find a first-order logic formula that is the negation of the original statement. Your final formula must not have any negations in it except for direct negations of predicates. For example, the negation of the formula $\forall x.\ (P(x) \rightarrow \exists y.\ (Q(x) \wedge R(y)))$ could be found by pushing the negation in from the outside inward as follows:

$$\neg(\forall x.\ (P(x) \rightarrow \exists y.\ (Q(x) \wedge R(y))))$$
$$\exists x.\ \neg(P(x) \rightarrow \exists y.\ (Q(x) \wedge R(y)))$$
$$\exists x.\ (P(x) \wedge \neg\exists y.\ (Q(x) \wedge R(y)))$$
$$\exists x.\ (P(x) \wedge \forall y.\ \neg(Q(x) \wedge R(y)))$$
$$\exists x.\ (P(x) \wedge \forall y.\ (Q(x) \rightarrow \neg R(y)))$$

Show every step of the process of pushing the negation into the formula (along the lines of what is done above). You don't need to formally prove that your negations are correct.

You may want to read over the Guide to Negations before attempting this problem.

    i. $\exists k.\ (RugbyPlayer(k) \wedge FootballPlayer(k) \wedge 49er(k))$

    ii. $\forall t.\ (Edible(t) \wedge Nutritious(t) \rightarrow Cultivated(t))$

    iii. $\forall p.\ \big(Person(p) \rightarrow (\exists q.\ (Person(q) \wedge TallerThan(p, q))) \vee (\exists q.\ (Person(q) \wedge TallerThan(q, p)))\big)$

    iv. $\exists r.\ (Silly(r) \leftrightarrow \neg Serious(r))$

Problem Seven: The Epimenides Paradox

Recall from Problem Set One that a paradox is a statement that leads to a contradiction regardless of whether it's true or false. One of the earliest paradoxes is the called the *Epimenides Paradox*, which is stated as follows:

Epimenides, a Cretan, says "All Cretans always lie."

According to the ancient Greeks, this statement is a paradox because Epimenides can neither tell the truth nor lie. A sketch of the argument is as follows:

> "If Epimenides tells the truth, then all Cretans always lie. Since Epimenides is himself a Cretan, then he must be lying, which is impossible because we know that Epimenides is telling the truth. This is a contradiction.
>
> If, on the other hand, Epimenides is lying, then his statement is false and all Cretans never lie. Since Epimenides himself is a Cretan, then he must be telling the truth, which is impossible because we know that he was lying. This too is a contradiction.
>
> Thus we have a contradiction regardless of whether Epimenides is lying or truthful – a paradox!"

However, there is a flaw in the above line of reasoning, and despite its name the Epimenides Paradox is ***not*** a paradox.

Identify the flaw in this reasoning. Since this is not really a paradox, Epimenides must either be lying or telling the truth. Is Epimenides lying or telling the truth? If he's telling the truth, why doesn't his statement contradict itself? If he's lying, why doesn't his statement contradict itself?

Problem Eight: Translating into Logic

Here are some statements we'd like you to translate into logic. As with the problem set, you should only use the provided predicates, functions, and constants in your translation.

We recommend reading over the Guide to Logic Translations before attempting this problem.

i.   Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "every cat is orange."

ii. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "some cat is orange."

iii. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "there are no orange cats."

iv. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "some cat is not orange."

v.   Given the predicates *Person*(*x*), which states that *x* is a person; *Orange*(*x*), which states that *x* is orange; *Cat*(*x*), which states that *x* is a cat; and *Likes*(*x*, *y*), which states that *x* likes *y*, write a formula in first-order logic that says "everyone likes at least one orange cat."

vi. Given the predicates *Person*(*x*), which states that x is a person; *Cat*(*x*), which states that *x* is a cat; and *Likes*(*x*, *y*), which states that x likes y, write a formula in first-order logic that says "everyone likes **exactly one** cat."

vii. Given the predicate *Person*(*x*), which states that *x* is a person, and *Muggle*(*x*), which states that *x* is a muggle, write a statement in first-order logic that says "some (but not all) people are muggles."

viii. Given the predicate *Person*(*x*), which states that *x* is a person, and *Ruler*(*x*), which states that *x* is a ruler, write a statement in first-order logic that says "there is at most one ruler."

ix. Given the predicate *Instant*(*i*), which states that *i* is an instant in time, and *Precedes*(*x*, *y*), which states that *x* precedes *y*, write a sentence in first-order logic that says "time has a beginning, but has no end."