Problems for Week Four

Problem One: Concept Checks

You know the drill. ☺ Here's a review from the topics from last week.

i. Give two examples of binary relations over the set \( \mathbb{N} \).

ii. What three properties must a binary relation have to have in order to be an equivalence relation? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation \( R \) has that property.

iii. If \( R \) is an equivalence relation over a set \( A \) and \( a \) is an element of \( A \), what does the notation \([a]_R\) mean? Intuitively, what does it represent?

iv. What three properties must a binary relation have to have in order to be a strict order? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation \( R \) has that property.

v. What is a Hasse diagram? Give an example.

vi. What does the notation \( f : A \rightarrow B \) mean?

vii. Let \( f : A \rightarrow B \) be a function. Express, in first-order logic, what property \( f \) has to satisfy to be an injection. Then, based on the structure of that formula, explain how you would write a proof that \( f \) is injective.

viii. Negate your statement from part (vii) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that \( f \) is not injective.

ix. Let \( f : A \rightarrow B \) be a function. Express, in first-order logic, what property \( f \) has to satisfy to be a surjection. Then, based on the structure of that formula, explain how you would write a proof that \( f \) is surjective.

x. Negate your statement from part (ix) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that \( f \) is not surjective.

xi. Let \( f : A \rightarrow B \) be a function. What properties must \( f \) have to be a bijection? How would you write a proof that \( f \) is bijective?

xii. What would you need to prove to show that \( f \) is not a bijection?
Problem Two: Equivalence Relations
This question explores various properties of equivalence relations.

i. In lecture, we proved that the binary relation ∼ over ℤ defined as follows is an equivalence relation:
   \[ a \sim b \text{ if } a+b \text{ is even.} \]
   Consider this new relation # defined over ℤ:
   \[ a \# b \text{ if } a+b \text{ is odd.} \]
   Is # an equivalence relation? If so, prove it. If not, disprove it.

ii. How many equivalence classes are there for the ∼ relation defined above? What are they?

Problem Three: Inverse Relations
Let \( R \) be a binary relation over a set \( A \). We can define a new relation over \( A \) called the inverse relation of \( R \), denoted \( R^{-1} \), as follows:
   \[ xR^{-1}y \text{ if } yRx \]
This question explores properties of inverse relations.

i. What is the inverse of the < relation over ℤ? Briefly justify your answer.

ii. What is the inverse of the = relation over ℤ? Briefly justify your answer.

iii. Prove or disprove: if \( R \) is an equivalence relation over \( A \), then \( R^{-1} \) is an equivalence relation over \( A \).

Problem Four: Monotone Functions
A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called monotone increasing if the following is true:
   \[ \forall x \in \mathbb{R}. \forall y \in \mathbb{R}. (x < y \rightarrow f(x) < f(y)) \]
This problem explores properties of monotone increasing functions.

i. Prove or disprove: every monotone increasing function is injective.

ii. Prove or disprove: every injective function from \( \mathbb{R} \) to \( \mathbb{R} \) is monotone increasing.

Problem Five: Involutions
A function \( f : A \rightarrow A \) is called an involution if \( f(f(x)) = x \) for all \( x \in A \).

i. Find three different examples of involutions from ℤ to ℤ. Briefly justify your answers.

ii. Prove that if \( f \) is an involution, then \( f \) is a bijection.

Problem Six: Functions and Relations – Together!
Let \( f : A \rightarrow B \) be an arbitrary function. Define a new binary relation ∼ over \( A \) as follows:
   \[ x \sim y \text{ if } f(x) = f(y) \]
Prove that ∼ is an equivalence relation over \( A \).