

Problems for Week Four

Problem One: Concept Checks

You know the drill. ☺ Here's a review from the topics from last week.

- i. Give two examples of binary relations over the set \mathbb{N} .
- ii. What three properties must a binary relation have to have in order to be an equivalence relation? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation R has that property.
- iii. If R is an equivalence relation over a set A and a is an element of A , what does the notation $[a]_R$ mean? Intuitively, what does it represent?
- iv. What three properties must a binary relation have to have in order to be a strict order? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation R has that property.
- v. What is a Hasse diagram? Give an example.
- vi. What does the notation $f: A \rightarrow B$ mean?
- vii. Let $f: A \rightarrow B$ be a function. Express, in first-order logic, what property f has to satisfy to be an injection. Then, based on the structure of that formula, explain how you would write a proof that f is injective.
- viii. Negate your statement from part (vii) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that f is not injective.
- ix. Let $f: A \rightarrow B$ be a function. Express, in first-order logic, what property f has to satisfy to be a surjection. Then, based on the structure of that formula, explain how you would write a proof that f is surjective.
- x. Negate your statement from part (ix) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that f is not surjective.
- xi. Let $f: A \rightarrow B$ be a function. What properties must f have to be a bijection? How would you write a proof that f is bijective?
- xii. What would you need to prove to show that f is not a bijection?

Problem Two: Equivalence Relations

This question explores various properties of equivalence relations.

- i. In lecture, we proved that the binary relation \sim over \mathbb{Z} defined as follows is an equivalence relation:

$$a \sim b \quad \text{if } a+b \text{ is even.}$$

Consider this new relation $\#$ defined over \mathbb{Z} :

$$a \# b \quad \text{if } a+b \text{ is odd.}$$

Is $\#$ an equivalence relation? If so, prove it. If not, disprove it.

- ii. How many equivalence classes are there for the \sim relation defined above? What are they?

Problem Three: Inverse Relations

Let R be a binary relation over a set A . We can define a new relation over A called the *inverse relation of R* , denoted R^{-1} , as follows:

$$xR^{-1}y \quad \text{if } yRx$$

This question explores properties of inverse relations.

- What is the inverse of the $<$ relation over \mathbb{Z} ? Briefly justify your answer.
- What is the inverse of the $=$ relation over \mathbb{Z} ? Briefly justify your answer.
- Prove or disprove: if R is an equivalence relation over A , then R^{-1} is an equivalence relation over A .

Problem Four: Monotone Functions

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *monotone increasing* if the following is true:

$$\forall x \in \mathbb{R}. \forall y \in \mathbb{R}. (x < y \rightarrow f(x) < f(y))$$

This problem explores properties of monotone increasing functions.

- Prove or disprove: every monotone increasing function is injective.
- Prove or disprove: every injective function from \mathbb{R} to \mathbb{R} is monotone increasing.

Problem Five: Involutions

A function $f: A \rightarrow A$ is called an *involution* if $f(f(x)) = x$ for all $x \in A$.

- Find three different examples of involutions from \mathbb{Z} to \mathbb{Z} . Briefly justify your answers.
- Prove that if f is an involution, then f is a bijection.

Problem Six: Functions and Relations – Together!

Let $f: A \rightarrow B$ be an arbitrary function. Define a new binary relation \sim over A as follows:

$$x \sim y \quad \text{if } f(x) = f(y)$$

Prove that \sim is an equivalence relation over A .