

## Problems for Week Five

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The problems contained in this packet consist of a review of the topics from last week (cardinality, graphs, and the pigeonhole principle), and feel free to work through whichever ones seem most interesting!

### Problem One: Cardinality Concept Checks

Here's some quick concept checks to make sure we're all on the same page.

- i. If  $A$  and  $B$  are sets, what is the formal definition of the statement  $|A| = |B|$ ?
- ii. If  $A$  and  $B$  are sets, what is the formal definition of the statement  $|A| \neq |B|$ ?
- iii. If  $A$  and  $B$  are sets, what is the formal definition of the statement  $|A| \leq |B|$ ?
- iv. If  $A$  and  $B$  are sets, what is the formal definition of the statement  $|A| < |B|$ ?

### Problem Two: Finding Functions

There are a lot of functions out there with interesting properties. This question asks you to think about different possible functions and what they might look like.

- i. Find a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  that is both injective and surjective. Prove it meets those criteria.
- ii. Find a function  $g: \mathbb{N} \rightarrow \mathbb{N}$  that is injective but not surjective. Prove it meets those criteria.
- iii. Find a function  $h: \mathbb{N} \rightarrow \mathbb{N}$  that is not injective but is surjective. Prove it meets those criteria.
- iv. Find a function  $k: \mathbb{N} \rightarrow \mathbb{N}$  that's neither injective nor surjective. Prove it meets those criteria.
- v. Based on your answers to these problems, explain why if you have a function  $m: A \rightarrow B$  and you know that  $|A| = |B|$ , you cannot necessarily say anything about whether  $m$  is injective, surjective, or bijective.

### Problem Three: $\aleph_0 \pm 1$

We use the symbol  $\aleph_0$  to denote the cardinality of the set  $\mathbb{N}$ . What does  $\aleph_0 - 1$  look like? This would be the cardinality of a set formed by starting with the natural numbers and deleting one of the elements.

- i. Briefly describe the set  $S$  in plain English.
- ii. Find a way of pairing elements of  $S$  with elements of  $\mathbb{N}$  so that no elements are uncovered.
- iii. Based on your answer to part (vi) of this problem, define a bijection  $f: S \rightarrow \mathbb{N}$ .
- iv. Prove that the function you found in part (iii) of this problem is a bijection. Since the cardinality of  $S$  is  $\aleph_0 - 1$ , this proves that  $\aleph_0 - 1 = \aleph_0$ .

The value  $\aleph_0 + 1$  is the cardinality of a set formed by starting with  $\mathbb{N}$  and adding in another value. So let's have  $T$  be the set  $\mathbb{N} \cup \{\star\}$ , where  $\star$  is some arbitrarily-chosen object that isn't a natural number.

- v. Briefly describe the set  $T$  in plain English.
- vi. Find a way of pairing elements of  $T$  with elements of  $\mathbb{N}$  so that no elements are uncovered.
- vii. Based on your answer to part (ii) of this problem, define a bijection  $g: T \rightarrow \mathbb{N}$ .
- viii. Prove that the function you found in part (iii) of this problem is a bijection. Since the cardinality of  $T$  is  $\aleph_0 + 1$ , this proves that  $\aleph_0 + 1 = \aleph_0$ .

### Problem Four: Graph Theory Concept Checks

Here's a quick review of our concepts from graph theory.

- i. What is an undirected graph? What is a directed graph?
- ii. What does it mean for two nodes to be adjacent in a graph?
- iii. What is a path in a graph? What is a simple path in a graph?
- iv. What is a cycle in a graph? What is a simple cycle in a graph?
- v. What does it mean for two nodes to be connected in a graph?
- vi. Is it possible for two nodes in a graph to be adjacent but not connected?
- vii. Is it possible for two nodes in a graph to be connected but not adjacent?
- viii. What does it mean for a graph  $G$  to be connected?
- ix. What is a connected component in a graph?
- x. How many connected components does each node in a graph belong to?
- xi. What is a planar graph?
- xii. What is a  $k$ -vertex-coloring of a graph?
- xiii. What does the notation  $\chi(G)$  mean?
- xiv. What is meant by the degree of a node in a graph?

### Problem Five: Graph Coloring

This problem explores some properties of graph coloring that may help you get a better intuitive feel for how colorings and node degrees relate to one another.

- i. Give an example of a 2-colorable graph where some node has degree seven. Briefly justify why your graph meets these criteria; no proof is necessary.
- ii. Generalize your answer from part (i) by describing how, for any  $n \geq 0$ , you can build a 2-colorable graph where some node has degree at least  $n$ . This shows that there is no direct connection between the maximum degree of a node in a graph and the chromatic number of that graph.
- iii. Give an example of a 2-colorable graph where every node has degree three. Briefly justify why your graph meets these criteria; no proof is necessary.
- iv. Generalize your answer from part (iii) by describing how, for any  $n \geq 0$ , you can build a 2-colorable graph where every node has degree at least  $n$ . This shows that there is no direct connection between the minimum degree of a node in the graph and the chromatic number of that graph.
- v. Give an example of a graph where every node has degree two but which is not 2-colorable. Briefly justify why your graph meets these criteria; no proof is necessary.
- vi. Generalize your answer from part (v) by describing how, for any  $n \geq 0$ , you can build a connected graph with at least  $n$  nodes where every node has degree two but which is not 2-colorable. This shows that each part of a graph can look 2-colorable even though the graph as a whole is not.

### Problem Six: The Pigeonhole Principle

Suppose you pick 11 numbers from the list 1, 2, 3, 4, 5, ..., 20. Prove that you must have chosen a pair of numbers whose difference is 10.