Problems for Week Eight

Problem One: Designing Regular Expressions

Below are a list of alphabets Σ and languages over those alphabets. For each language, write a regular expression for that language.

i. Let Σ = {a, b, c} and let L = { $w \in \Sigma^*$ | w ends in cab }. Write a regular expression for L.

ii. Let Σ = {a, b} and let L = { $w \in \Sigma^*$ | $w \neq \varepsilon$ and the first and last character of w are the same }. Write a regular expression for L.

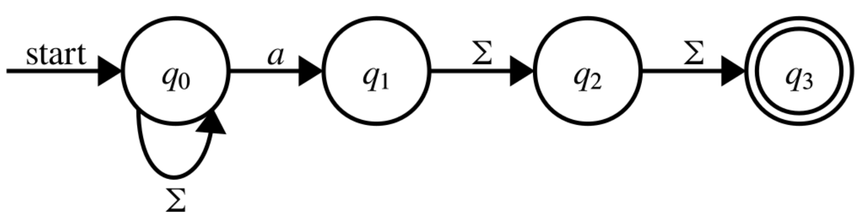
iii. Let Σ = {a, b} and let L = { $w \in \Sigma^*$ | w contains two b's separated by exactly five characters }. Write a regular expression for L.

iv. Let Σ = {a, b} and let L = { $w \in \Sigma^*$ | w is a nonempty string whose characters alternate between a's and b's }. Write a regular expression for L.

v. Let Σ = {a, b, c} and let L = { $w \in \Sigma^*$ | w contains every character in Σ exactly once }. Write a regular expression for L.

Problem Two: State Elimination

Below is an NFA for a language from last week's packet of problems:



Using the state-elimination algorithm, convert this NFA into a regular expression. (You could just directly design a regular expression for this language, but we want you to specifically use the state elimination algorithm).

Problem Three: The Myhill-Nerode Theorem

The Myhill-Nerode theorem says the following:

> Let L be a language over Σ. If there is a set $S \subseteq \Sigma^*$ such that
>
> · S contains infinitely many strings, and
>
> · any two distinct strings $x, y \in S$ are distinguishable relative to L (that is, $x \not\equiv_L y$),
>
> then L is not a regular language.

Below is a (slightly modified) version of the proof of the Myhill-Nerode theorem from lecture:

> ***Proof:*** Let L be an arbitrary language over Σ. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that L is not regular.
>
> Suppose for the sake of contradiction that L is regular. This means that there must be some DFA D for L. Let k be the number of states in D. Since S is an infinite set, we can choose $k+1$ distinct strings from S and run each of those strings through D. Because there are only k states in D and we've chosen $k+1$ distinct strings from S, by the pigeonhole principle we know that at least two strings from S must end in the same state in D. Choose any two such strings and call them x and y.
>
> Since $x \in S$ and $y \in S$ and $x \neq y$, we know that $x \not\equiv_L y$. Consequently, by our earlier theorem, we know that x and y must end in different states when run through D. But this is impossible – we chose x and y specifically because they end in the same state when run through D. We have reached a contradiction, so our assumption must have been wrong. Thus L is not a regular language. ■

This question explores the theorem in a bit more detail.

i. What is the formal definition of the statement $x \not\equiv_L y$? Explain it in plain English. Give an example of two strings x and y along with a language L where $x \not\equiv_L y$ holds.

ii. The proof hinges on the fact that if $x \not\equiv_L y$, then x and y cannot end in the same state when run through any DFA for a language L. We sketched a proof of this in class. Explain intuitively why this is the case.

iii. Explain, intuitively, why S has to be an infinite set for this proof to work.

iv. Does anything in the proof require that S be a subset of L?

Problem Four: Nonregular Languages Warmup

Let $\Sigma = \{1, \geq\}$ and consider the language $L = \{1^m{\geq}1^n \mid m, n \in \mathbb{N} \text{ and } m \geq n\}$.

i. Give some specific examples of strings from the language L.

ii. Without using the Myhill-Nerode theorem, give an intuitive justification for why L isn't regular.

iii. Use the Myhill-Nerode theorem to prove that L isn't regular. You'll need to find an infinite set of strings that are pairwise distinguishable relative to L. As a hint, see if you can think of some strings that would have to be treated differently by any DFA for L, then see what happens if you gather all of them together into a set.

Problem Five: Nonregular Languages

Here are some more problems to help you get used to proving that certain languages aren't regular.

i. Let $\Sigma = \{a, b\}$ and let L = $\{ a^n b^m \mid n, m \in \mathbb{N} \text{ and } n \neq m \}$. Explain why this language is not the complement of the language $\{ a^n b^n \mid n \in \mathbb{N} \}$.

ii. Let $\Sigma = \{a, b\}$ and let $L = \{ a^n b^m \mid n, m \in \mathbb{N} \text{ and } n \neq m \}$. Prove that L is not regular.

iii. Let $\Sigma = \{a\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ is a palindrome} \}$. Prove that L *is* regular.

iv. Let $\Sigma = \{a, b\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ is a palindrome} \}$. Prove that L is not regular.