Problems for Week Eight

Problem One: Designing Regular Expressions

Below are a list of alphabets \( \Sigma \) and languages over those alphabets. For each language, write a regular expression for that language.

i. Let \( \Sigma = \{ a, b, c \} \) and let \( L = \{ w \in \Sigma^* \mid w \text{ ends in } cab \} \). Write a regular expression for \( L \).

ii. Let \( \Sigma = \{ a, b \} \) and let \( L = \{ w \in \Sigma^* \mid w \neq \epsilon \text{ and the first and last character of } w \text{ are the same} \} \). Write a regular expression for \( L \).

iii. Let \( \Sigma = \{ a, b \} \) and let \( L = \{ w \in \Sigma^* \mid w \text{ contains two } b \text{'s separated by exactly five characters} \} \). Write a regular expression for \( L \).

iv. Let \( \Sigma = \{ a, b \} \) and let \( L = \{ w \in \Sigma^* \mid w \text{ is a nonempty string whose characters alternate between } a \text{'s and } b \text{'s} \} \). Write a regular expression for \( L \).

v. Let \( \Sigma = \{ a, b, c \} \) and let \( L = \{ w \in \Sigma^* \mid w \text{ contains every character in } \Sigma \text{ exactly once} \} \). Write a regular expression for \( L \).

Problem Two: State Elimination

Below is an NFA for a language from last week's packet of problems:

![NFA Diagram]

Using the state-elimination algorithm, convert this NFA into a regular expression. (You could just directly design a regular expression for this language, but we want you to specifically use the state elimination algorithm).
Problem Three: The Myhill-Nerode Theorem

The Myhill-Nerode theorem says the following:

Let $L$ be a language over $\Sigma$. If there is a set $S \subseteq \Sigma^*$ such that

- $S$ contains infinitely many strings, and
- any two distinct strings $x, y \in S$ are distinguishable relative to $L$ (that is, $x \not\equiv_L y$),

then $L$ is not a regular language.

Below is a (slightly modified) version of the proof of the Myhill-Nerode theorem from lecture:

**Proof:** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

Suppose for the sake of contradiction that $L$ is regular. This means that there must be some DFA $D$ for $L$. Let $k$ be the number of states in $D$. Since $S$ is an infinite set, we can choose $k+1$ distinct strings from $S$ and run each of those strings through $D$. Because there are only $k$ states in $D$ and we've chosen $k+1$ distinct strings from $S$, by the pigeonhole principle we know that at least two strings from $S$ must end in the same state in $D$. Choose any two such strings and call them $x$ and $y$.

Since $x \in S$ and $y \in S$ and $x \neq y$, we know that $x \not\equiv_L y$. Consequently, by our earlier theorem, we know that $x$ and $y$ must end in different states when run through $D$. But this is impossible – we chose $x$ and $y$ specifically because they end in the same state when run through $D$. We have reached a contradiction, so our assumption must have been wrong. Thus $L$ is not a regular language. ■

This question explores the theorem in a bit more detail.

i. What is the formal definition of the statement $x \not\equiv_L y$? Explain it in plain English. Give an example of two strings $x$ and $y$ along with a language $L$ where $x \not\equiv_L y$ holds.

ii. The proof hinges on the fact that if $x \not\equiv_L y$, then $x$ and $y$ cannot end in the same state when run through any DFA for a language $L$. We sketched a proof of this in class. Explain intuitively why this is the case.

iii. Explain, intuitively, why $S$ has to be an infinite set for this proof to work.

iv. Does anything in the proof require that $S$ be a subset of $L$?
Problem Four: Nonregular Languages Warmup

Let $\Sigma = \{1, \geq\}$ and consider the language $L = \{1^n \geq 1^m \mid m, n \in \mathbb{N} \text{ and } m \geq n \}$. 

i. Give some specific examples of strings from the language $L$.

ii. Without using the Myhill-Nerode theorem, give an intuitive justification for why $L$ isn't regular.

iii. Use the Myhill-Nerode theorem to prove that $L$ isn't regular. You'll need to find an infinite set of strings that are pairwise distinguishable relative to $L$. As a hint, see if you can think of some strings that would have to be treated differently by any DFA for $L$, then see what happens if you gather all of them together into a set.

Problem Five: Nonregular Languages

Here are some more problems to help you get used to proving that certain languages aren't regular.

i. Let $\Sigma = \{a, b\}$ and let $L = \{a^n b^m \mid n, m \in \mathbb{N} \text{ and } n \neq m \}$. Explain why this language is not the complement of the language $\{a^n b^n \mid n \in \mathbb{N}\}$.

ii. Let $\Sigma = \{a, b\}$ and let $L = \{a^n b^m \mid n, m \in \mathbb{N} \text{ and } n \neq m \}$. Prove that $L$ is not regular.

iii. Let $\Sigma = \{a\}$ and let $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$. Prove that $L$ is regular.

iv. Let $\Sigma = \{a, b\}$ and let $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$. Prove that $L$ is not regular.