Mathematical Logic
Where We Are Now

- Our coverage of logic focused on these topics:
  - Propositional variables.
  - Propositional connectives.
  - Propositional equivalences.
  - Predicates, functions, and constant symbols.
  - Objects and propositions.
  - Quantifiers.
  - Evaluating first-order formulas relative to a world.
  - Translating into first-order logic.
  - Negating and simplifying first-order formulas.
- Your goal this week is to keep your proof skills sharp while mastering the ins and outs of first-order logic.
Things You Should Do Today

● Review the solutions for Problem Set One and make sure you completely and unambiguously understand the answers. Ask for help if this isn't the case!

● Read the “Guide to Negating Formulas” and “Guide to First-Order Translation” on the course website to get more exposure and practice with those skills.

● Continue working through PS2.
Things You Should Do Tomorrow

- Look over your feedback on PS1 and make sure you understand all the feedback you get *completely* and *unambiguously*. Ask the course staff for help, either on Piazza or in office hours, if you don't.

- Continue working on PS2. If at all possible, aim to complete two of Q8, Q9, and Q10 and start writing up your answers.

- Start reviewing your partner's answers and compiling a single, definitive set of answers that you're going to turn in.

- Stop by office hours to get feedback on your proofs and take that feedback seriously.
Mechanics: *Negating Statements*
∀p. (Person(p) \rightarrow
    \exists q. (Person(q) \land p \neq q \land
    Loves(p, q)
) )
\neg \forall p. (\text{Person}(p) \rightarrow \\
\exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q))
)
\[ \neg \forall p. (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \]
\neg \forall p. (\text{Person}(p) \rightarrow \\
\exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q) \\
) \\
) \\
)
\[ \neg \forall p. (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \) \]
\neg \forall p. \ (Person(p) \rightarrow
\exists q. \ (Person(q) \land p \neq q \land
Loves(p, q))
)
)

\neg \forall x. A
\hline
\exists x. \neg A
\[ \exists p. \neg (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \) \]

\[ \neg \forall x. A \]

\[ \exists x. \neg A \]
\[ \exists p. \neg (\text{Person}(p) \rightarrow \\
\exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q)) \\
) \]
\[ \exists p. \neg (Person(p) \to \\
\exists q. (Person(q) \land p \neq q \land \\
Loves(p, q) \)) 
\]

\[
\begin{align*}
\neg(A \to B) \\
\hline
A \land \neg B
\end{align*}
\]
\[ \exists p. \neg (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \) \]
\[ \exists p. \neg (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q))) \]

\[ \neg (A \rightarrow B) \]

\[ A \land \neg B \]
\[ \exists p. (\text{Person}(p) \land \\
\neg \exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q) \\
) \\
) \\
\neg (A \rightarrow B) \\
\hline \\
A \land \neg B \]
$\exists p. (\text{Person}(p) \land \\
\neg \exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q) \\
))$

$\exists p. (\text{Person}(p) \land \\
\neg \exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q) \\
))$
\[\exists p. (\text{Person}(p) \land \neg \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \])\]
\[\exists p. \ (\text{Person}(p) \land \\
\neg \exists q. \ (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q)) \) \]
\[ \exists p. \ (\text{Person}(p) \land \neg \exists q. \ (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \) \]
\( \exists p. (\text{Person}(p) \land \forall q. \neg (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \) )

\[\neg \exists x. A \quad \underline{\therefore} \quad \forall x. \neg A\]
\( \exists p. \ (\text{Person}(p) \land \\
\forall q. \neg (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q)) \\
) \\
) \)
\[ \exists p. (\text{Person}(p) \land \forall q. \neg (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \]

\[ \neg (A \land B) \]

\[ A \rightarrow \neg B \]
\[ \exists p. \ (\text{Person}(p) \land \\
\forall q. \neg(\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q)) \\
) \]

\[ \neg(A \land B) \]

\[ A \rightarrow \neg B \]
\(\exists p. (\text{Person}(p) \land \forall q. \neg(\text{Person}(q) \land p \neq q \land \text{Loves}(p, q))\)

\(\neg(A \land B)\)

\(A \rightarrow \neg B\)
\[ \exists p. \ (\text{Person}(p) \land \forall q. \ (\text{Person}(q) \land p \neq q \rightarrow \neg \text{Loves}(p, q)) \) \]

\[ \neg(A \land B) \]

\[ A \rightarrow \neg B \]
\[ \exists p. (\text{Person}(p) \land \\
\forall q. (\text{Person}(q) \land p \neq q \rightarrow \\
\neg \text{Loves}(p, q) \\
) \) \\
) \]
∀p. (Person(p) →
    ∃q. (Person(q) ∧ p ≠ q ∧ Loves(p, q))
)
)

∃p. (Person(p) ∧
    ∀q. (Person(q) ∧ p ≠ q →
        ¬Loves(p, q))
)
)
∃p. (Person(p) ∧
    ∀q. (Person(q) ∧ p ≠ q → Loves(q, p))
)

Your turn!

Try negating this formula with the other folks at your table. See what you come up with!
\neg \exists p. (\text{Person}(p) \land \\
\forall q. (\text{Person}(q) \land p \neq q \rightarrow \\
\text{Loves}(q, p) \\
\)) \land \\
\)
\[\forall p. \neg (Person(p) \land \forall q. (Person(q) \land p \neq q \rightarrow Loves(q, p)) \)\]
∀p. (Person(p) →
   ¬∀q. (Person(q) ∧ p ≠ q → Loves(q, p)
   )
 )
\( \forall p. \ (Person(p) \rightarrow \) \\
\( \exists q. \ \neg(Person(q) \land p \neq q \rightarrow Loves(q, p) \) \\
\) \\
\)
∀p. (Person(p) → \\
∃q. (Person(q) \land p \neq q \land \\
\neg Loves(q, p)
)
)
\[\exists p. (\text{Person}(p) \wedge \\
\quad \forall q. (\text{Person}(q) \wedge p \neq q \rightarrow \\
\quad \quad \text{Loves}(q, p))\]

\[\forall p. (\text{Person}(p) \rightarrow \\
\quad \exists q. (\text{Person}(q) \wedge p \neq q \wedge \\
\quad \quad \neg \text{Loves}(q, p))\]
Techniques: *Translating Statements*
Common Patterns

• A statement of the form
  \( \forall x. (P(x) \rightarrow Q(x)) \)
  can be read as “all P's are Q's.”

• A statement of the form
  \( \exists x. (P(x) \land Q(x)) \)
  can be read as “there is a P that is also a Q” or “some P's are Q's.”

• **Remember:** If you see \( \exists \) paired with \( \rightarrow \) or \( \forall \) paired with \( \land \), the statement is probably incorrect!
Given the predicates

- \(\text{Person}(p)\), which states that \(p\) is a person, and
- \(\text{CanLearnFrom}(x, y)\), which says that \(x\) can learn from \(y\),

write a statement in first-order logic that says “everyone has someone they can learn from.”
Everyone has someone they can learn from
Every person \( p \) has someone they can learn from
Every person $p$ has someone they can learn from

“All As are Bs.”

$\forall x. (A(x) \rightarrow B(x))$
∀p. (Person(p) →
    p has someone they can learn from
)
\[ \forall p. \ (\text{Person}(p) \rightarrow \text{there is a person } q \text{ that } p \text{ can learn from}) \]
∀p. (\( \text{Person}(p) \rightarrow \text{there is a person q that p can learn from} \))
∀p. (Person(p) →
   ∃q. (Person(q) ∧
       p can learn from q)
  )
)
∀p. (Person(p) →
   ∃q. (Person(q) ∧
       CanLearnFrom(p, q)
   )
)
Consider this statement:

“If someone is happy, then everyone is happy.”

What is the contrapositive of this statement?
If someone is happy, then everyone is happy
someone is happy $\rightarrow$ everyone is happy
someone is happy $\rightarrow (\forall x. \text{Happy}(x))$
(∃x. Happy(x)) → (∀x. Happy(x))
(∃x. Happy(x)) → (∀x. Happy(x))
(∃x. Happy(x)) \rightarrow (∀x. Happy(x))
\[ \neg(\forall x. \text{Happy}(x)) \rightarrow \neg(\exists x. \text{Happy}(x)) \]
$(\exists x. \neg \text{Happy}(x)) \rightarrow \neg (\exists x. \text{Happy}(x))$
$(\exists x. \neg Happy(x)) \rightarrow (\forall x. \neg Happy(x))$
(∃x. ¬Happy(x)) → (∀x. ¬Happy(x))

“If someone is not happy, then everyone is not happy.”
Consider this statement:

“If someone is happy, then everyone is happy.”

What is the negation of this statement?
(∃x. Happy(x)) → (∀x. Happy(x))
$(\exists x. \text{Happy}(x)) \rightarrow (\forall x. \text{Happy}(x))$

$\neg((\exists x. \text{Happy}(x)) \rightarrow (\forall x. \text{Happy}(x)))$
$(\exists x. \text{Happy}(x)) \rightarrow (\forall x. \text{Happy}(x))$

$(\exists x. \text{Happy}(x)) \land \neg(\forall x. \text{Happy}(x))$
$$\exists x. \text{Happy}(x) \rightarrow \forall x. \text{Happy}(x)$$

$$\exists x. \text{Happy}(x) \land \exists x. \neg \text{Happy}(x)$$
(∃x. Happy(x)) → (∀x. Happy(x))

(∃x. Happy(x)) ∧ (∃x. ¬Happy(x))

“Someone is happy and someone is not happy.”