

Mathematical Logic

Where We Are Now

- Week 2 covered these key topics:
 - Propositional variables.
 - Propositional connectives.
 - Propositional equivalences.
 - Predicates, functions, and constant symbols.
 - Objects and propositions.
 - Quantifiers.
 - Evaluating first-order formulas relative to a world.
 - Translating into first-order logic.
 - Negating and simplifying first-order formulas.
- Your goal this week is to keep your proof skills sharp while mastering the ins and outs of first-order logic.

Where We're Going

- Week 3 is about *discrete structures*:
 - Binary relations. (Yesterday)
 - Equivalence relations. (Yesterday/Tomorrow)
 - Strict order relations (Tomorrow)
 - Functions (Friday)
- From yesterday, you should know what a binary relation is, what the terms *reflexive*, *symmetric*, and *transitive* mean, and how to prove that a binary relation has those properties.

Things You Should Do Today

- Review the checkpoint problem from Problem Set Two and make sure you *completely* and *unambiguously* understand the answers. Ask for help if this isn't the case!
- Read the “Guide to Negating Formulas” and “Guide to First-Order Translation” on the course website to get more exposure and practice with those skills.
- Continue working through PS2. Aim to complete Q1 – Q6 by tonight, if possible.

Things You Should Do Tomorrow

- Look over your feedback on PS1 and make sure you understand all the feedback you get **completely** and **unambiguously**. Ask the course staff for help, either on Piazza or in office hours, if you don't.
- Continue working on PS2. If at all possible, aim to complete two of Q7, Q8, and Q9 and start writing up your answers.
- Start reviewing your partner's answers and compiling a single, definitive set of answers that you're going to turn in.
- Stop by office hours to get feedback on your proofs and take that feedback seriously.

Mechanics: *Negating Statements*

$$\forall p. (Person(p) \rightarrow$$
$$\quad \exists q. (Person(q) \wedge p \neq q \wedge$$
$$\quad \quad Loves(p, q)$$
$$\quad)$$
$$)$$

$$\neg \forall p. (Person(p) \rightarrow$$
$$\exists q. (Person(q) \wedge p \neq q \wedge$$
$$Loves(p, q)$$
$$)$$
$$)$$

$\neg \forall p. (Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

Useful Resource:

Go to cs103.stanford.edu and
read the Guide to Negating
Formulas.

$\neg \forall p. (Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
 $)$
 $)$

$$\frac{\neg \forall x. A}{\exists x. \neg A}$$

$\neg \forall p. (Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg \forall x. A}{\exists x. \neg A}$$

$\neg \forall p. (Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg \forall x. A}{\exists x. \neg A}$$

$\exists p. \neg(Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
 $)$
 $)$

$$\frac{\neg \forall x. A}{\exists x. \neg A}$$

$$\begin{aligned} & \exists p. \neg(Person(p) \rightarrow \\ & \quad \exists q. (Person(q) \wedge p \neq q \wedge \\ & \quad \quad Loves(p, q) \\ & \quad) \\ &) \end{aligned}$$

$\exists p. \neg(Person(p) \rightarrow$
 $\quad \exists q. (Person(q) \wedge p \neq q \wedge$
 $\quad \quad Loves(p, q)$
 $\quad)$
 $)$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$\exists p. \neg(Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$\exists p. \neg(Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$\exists p. (Person(p) \wedge$
 $\neg \exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\exists p. (Person(p) \wedge$$
$$\neg \exists q. (Person(q) \wedge p \neq q \wedge$$
$$Loves(p, q)$$
$$)$$
$$)$$

$\exists p. (Person(p) \wedge$
 $\neg \exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg \exists x. A}{\forall x. \neg A}$$

$\exists p. (Person(p) \wedge$
 $\neg \exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg \exists x. A}{\forall x. \neg A}$$

$\exists p. (Person(p) \wedge$
 $\neg \exists q. (Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg \exists x. A}{\forall x. \neg A}$$

$\exists p. (Person(p) \wedge$
 $\forall q. \neg(Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
 $)$
 $)$

$$\frac{\neg \exists x. A}{\forall x. \neg A}$$

$$\begin{aligned} & \exists p. (Person(p) \wedge \\ & \quad \forall q. \neg(Person(q) \wedge p \neq q \wedge \\ & \quad \quad Loves(p, q) \\ & \quad) \\ &) \end{aligned}$$

$\exists p. (Person(p) \wedge$
 $\quad \forall q. \neg(Person(q) \wedge p \neq q \wedge$
 $\quad \quad Loves(p, q)$
 $\quad)$
 $)$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$\exists p. (Person(p) \wedge$
 $\quad \forall q. \neg(Person(q) \wedge p \neq q \wedge$
 $\quad \quad Loves(p, q)$
 $\quad)$
 $)$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$\exists p. (Person(p) \wedge$
 $\forall q. \neg(Person(q) \wedge p \neq q \wedge$
 $Loves(p, q)$
)
)

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$\exists p. (Person(p) \wedge$
 $\forall q. (Person(q) \wedge p \neq q \rightarrow$
 $\neg Loves(p, q)$
)
)

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\begin{aligned} & \exists p. (Person(p) \wedge \\ & \quad \forall q. (Person(q) \wedge p \neq q \rightarrow \\ & \quad \quad \neg Loves(p, q) \\ & \quad) \\ &) \end{aligned}$$

$$\forall p. (Person(p) \rightarrow$$
$$\quad \exists q. (Person(q) \wedge p \neq q \wedge$$
$$\quad \quad Loves(p, q)$$
$$\quad)$$
$$)$$
$$\exists p. (Person(p) \wedge$$
$$\quad \forall q. (Person(q) \wedge p \neq q \rightarrow$$
$$\quad \quad \neg Loves(p, q)$$
$$\quad)$$
$$)$$

$$\begin{aligned} & \exists p. (Person(p) \wedge \\ & \quad \forall q. (Person(q) \wedge p \neq q \rightarrow \\ & \quad \quad Loves(q, p) \\ & \quad) \\ &) \end{aligned}$$

Your turn!

Try negating this formula with
the other folks at your table.
See what you come up with!

$$\neg \exists p. (Person(p) \wedge$$
$$\forall q. (Person(q) \wedge p \neq q \rightarrow$$
$$Loves(q, p)$$
$$)$$
$$)$$

$$\forall p. \neg(Person(p) \wedge$$
$$\forall q. (Person(q) \wedge p \neq q \rightarrow$$
$$Loves(q, p)$$
$$)$$
$$)$$

$$\forall p. (Person(p) \rightarrow$$
$$\neg \forall q. (Person(q) \wedge p \neq q \rightarrow$$
$$Loves(q, p)$$
$$)$$
$$)$$

$$\forall p. (Person(p) \rightarrow$$
$$\quad \exists q. \neg(Person(q) \wedge p \neq q \rightarrow$$
$$\quad \quad Loves(q, p)$$
$$\quad)$$
$$)$$

$$\forall p. (Person(p) \rightarrow$$
$$\quad \exists q. (Person(q) \wedge p \neq q \wedge$$
$$\quad \quad \neg Loves(q, p)$$
$$\quad)$$
$$)$$

$$\begin{aligned} & \exists p. (Person(p) \wedge \\ & \quad \forall q. (Person(q) \wedge p \neq q \rightarrow \\ & \quad \quad Loves(q, p) \\ & \quad) \\ &) \end{aligned}$$
$$\begin{aligned} & \forall p. (Person(p) \rightarrow \\ & \quad \exists q. (Person(q) \wedge p \neq q \wedge \\ & \quad \quad \neg Loves(q, p) \\ & \quad) \\ &) \end{aligned}$$

Techniques: ***Translating Statements***

Common Patterns

- A statement of the form

$$\forall x. (P(x) \rightarrow Q(x))$$

can be read as “all P 's are Q 's.”

- A statement of the form

$$\exists x. (P(x) \wedge Q(x))$$

can be read as “there is a P that is also a Q ” or “some P 's are Q 's.”

- **Remember:** If you see \exists paired with \rightarrow or \forall paired with \wedge , the statement is probably incorrect!

Given the predicates

- *Person*(p), which states that p is a person, and
- *CanLearnFrom*(x, y), which says that x can learn from y ,

write a statement in first-order logic that says “everyone has someone they can learn from.”

Everyone has someone they can learn from

Every person p has someone they can learn from

Every person p has someone they can learn from

“All A s are B s.”
 $\forall x. (A(x) \rightarrow B(x))$

$\forall p. (Person(p) \rightarrow$
p has someone they can learn from
)

$\forall p. (Person(p) \rightarrow$
there is a person q that p can learn from
)

$\forall p. (\text{Person}(p) \rightarrow$
there is a person q that p can learn from
)

“Some As are Bs.”

$\exists x. (A(x) \wedge B(x))$

$\forall p. (Person(p) \rightarrow$
 $\exists q. (Person(q) \wedge$
 p can learn from q
)
)

$$\forall p. (Person(p) \rightarrow$$
$$\quad \exists q. (Person(q) \wedge$$
$$\quad \quad CanLearnFrom(p, q)$$
$$\quad)$$
$$)$$

Consider this statement:

“If someone is happy, then everyone is happy.”

What is the *contrapositive* of this statement?

If someone is happy, then everyone is happy

someone is happy → *everyone is happy*

someone is happy $\rightarrow (\forall x. \text{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$$\neg(\forall x. \textit{Happy}(x)) \rightarrow \neg(\exists x. \textit{Happy}(x))$$

$$(\exists x. \neg \textit{Happy}(x)) \rightarrow \neg(\exists x. \textit{Happy}(x))$$

$$(\exists x. \neg \textit{Happy}(x)) \rightarrow (\forall x. \neg \textit{Happy}(x))$$

$$(\exists x. \neg \text{Happy}(x)) \rightarrow (\forall x. \neg \text{Happy}(x))$$

“If someone is not happy, then everyone is not happy.”

Consider this statement:

“If someone is happy, then everyone is happy.”

What is the *negation* of this statement?

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$\neg((\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x)))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \wedge \neg(\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \wedge (\exists x. \neg \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \rightarrow (\forall x. \textit{Happy}(x))$

$(\exists x. \textit{Happy}(x)) \wedge (\exists x. \neg \textit{Happy}(x))$

“Someone is happy and someone is not happy.”