Mathematical Logic

Where We Are Now

- Our coverage of logic focused on these topics:
  - Propositional variables.
  - Propositional connectives.
  - Propositional equivalences.
  - Predicates, functions, and constant symbols.
  - Objects and propositions.
  - Quantifiers.
  - Evaluating first-order formulas relative to a world.
  - Translating into first-order logic.
  - Negating and simplifying first-order formulas.
- Your goal this week is to keep your proof skills sharp while mastering the ins and outs of first-order logic.

Things You Should Do Today

- Review the solutions for Problem Set One and make sure you *completely* and *unambiguously* understand the answers. Ask for help if this isn't the case!
- Read the “Guide to Negating Formulas” and “Guide to First-Order Translation” on the course website to get more exposure and practice with those skills.
- Continue working through PS2.

Things You Should Do Tomorrow

- Look over your feedback on PS1 and make sure you understand all the feedback you get ***completely*** and ***unambiguously***. Ask the course staff for help, either on Piazza or in office hours, if you don't.
- Continue working on PS2. If at all possible, aim to complete two of Q8, Q9, and Q10 and start writing up your answers.
- Start reviewing your partner's answers and compiling a single, definitive set of answers that you're going to turn in.
- Stop by office hours to get feedback on your proofs and take that feedback seriously.

Mechanics: ***Negating Statements***

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Useful Resource:

Go to cs103.stanford.edu and read the Guide to Negating Formulas.

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}$$

$$\neg\forall p.\ (Person(p) \rightarrow$$
$$\quad \exists q.\ (Person(q) \land p \neq q \land$$
$$\quad\quad Loves(p, q)$$
$$\quad)$$
$$)$$

$$\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}$$

$$
\begin{aligned}
&\neg \forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg \forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{array}{l}
\exists p.\ \neg(Person(p) \to \\
\quad \exists q.\ (Person(q) \land p \neq q \land \\
\quad\quad Loves(p, q) \\
\quad) \\
)
\end{array}
$$

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&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg(A \rightarrow B)}{A \land \neg B}}$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg(A \rightarrow B)}{A \land \neg B}}$$

$$
\begin{aligned}
&\exists p.\ \neg(\mathit{Person}(p) \to \\
&\quad \exists q.\ (\mathit{Person}(q) \land p \neq q \land \\
&\qquad \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \to B)}{A \land \neg B}}
$$

$\exists p.\ (Person(p)\ \land$
$\quad \neg\exists q.\ (Person(q) \land p \neq q \land$
$\quad\quad Loves(p, q)$
$\quad)$
$)$

$$\frac{\neg(A \to B)}{A \land \neg B}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \neg\exists q.\ (Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \neg\exists q.\ (Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \neg\exists q.\ (Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}$$

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\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \neg\exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg(Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg\exists x.\ A}{\forall x.\ \neg A}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg(Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg(Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg(A \land B)}{A \to \neg B}}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg(Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg({\color{red}A} \land {\color{blue}B})}{{\color{red}A} \to \neg{\color{blue}B}}$$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ \neg(Person(q) \land p \neq q \land$

$\qquad Loves(p, q)$

$\quad)$

$)$

$$\frac{\neg(A \land B)}{A \to \neg B}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad \neg Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg(A \land B)}{A \rightarrow \neg B}}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\qquad \neg Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\forall p.\ (Person(p) \to \\
\quad \exists q.\ (Person(q) \land p \neq q \land \\
\quad\quad Loves(p, q) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists p.\ (Person(p) \land \\
\quad \forall q.\ (Person(q) \land p \neq q \to \\
\quad\quad \neg Loves(p, q) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \to \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Your turn!

Try negating this formula with the other folks at your table. See what you come up with!

$$
\begin{aligned}
&\neg\exists p.\ (\mathit{Person}(p)\ \land \\
&\quad \forall q.\ (\mathit{Person}(q)\ \land\ p \neq q \rightarrow \\
&\qquad \mathit{Loves}(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ \neg(Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\forall p.\ (Person(p) \rightarrow \\
\quad \neg\forall q.\ (Person(q) \land p \neq q \rightarrow \\
\qquad Loves(q, p) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ \neg(Person(q) \land p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad \neg Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \to \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q)\ \land\ p \neq q\ \land \\
&\qquad \neg Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Techniques: ***Translating Statements***

Common Patterns

- A statement of the form

$$\forall x.\ (P(x) \to Q(x))$$

  can be read as "all P's are Q's."

- A statement of the form

$$\exists x.\ (P(x) \land Q(x))$$

  can be read as "there is a P that is also a Q" or "some P's are Q's."

- ***Remember:*** If you see $\exists$ paired with $\to$ or $\forall$ paired with $\land$, the statement is probably incorrect!

Given the predicates

- $Person(p)$, which states that p is a person, and
- $CanLearnFrom(x, y)$, which says that x can learn from y,

write a statement in first-order logic that says "everyone has someone they can learn from."

Everyone has someone they can learn from

Every person p has someone they can learn from

Every person p has someone they can learn from

"All As are Bs."

$$\forall x.\ (A(x) \to B(x))$$

$\forall p.\ (Person(p) \rightarrow$

p has someone they can learn from

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q that p can learn from

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q that p can learn from

$)$

“Some *A*s are *B*s.”

$$\exists x.\ (A(x) \land B(x))$$

$\forall p. \; (Person(p) \rightarrow$

$\quad \exists q. \; (Person(q) \; \land$

$\quad\quad$ *p can learn from q*

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q)\ \wedge \\
&\qquad CanLearnFrom(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Consider this statement:

"If someone is happy, then everyone is happy."

What is the *contrapositive* of this statement?

If someone is happy, then everyone is happy *someone is happy* → *everyone is happy* *someone is happy* $\to (\forall x.\ Happy(x))$

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$\neg(\forall x.\ Happy(x)) \rightarrow \neg(\exists x.\ Happy(x))$$

$$(\exists x.\ \neg Happy(x)) \rightarrow \neg(\exists x.\ Happy(x))$$

$$(\exists x.\ \neg Happy(x)) \rightarrow (\forall x.\ \neg Happy(x))$$

$$(\exists x.\ \neg Happy(x)) \to (\forall x.\ \neg Happy(x))$$

"If someone is not happy, then everyone is not happy."

Consider this statement:

"If someone is happy, then everyone is happy."

What is the *negation* of this statement?

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$\neg((\exists x.\ Happy(x)) \to (\forall x.\ Happy(x)))$$

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land \neg(\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land (\exists x.\ \neg Happy(x))$$

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land (\exists x.\ \neg Happy(x))$$

“Someone is happy and someone is not happy.”