CS 103X: Discrete Structures
Homework Assignment 1

Due January 20, 2006

Exercise 1 (Reading Assignment)**.** Read pages 15–20 in Lehman-Leighton.

Exercise 2. Let A be a set containing n elements. For each of the following sets, state its cardinality. If more information is needed in order to answer, explain what is missing.

(a) $A \cup \emptyset$

(b) $A \cap \emptyset$

(c) $A \cup \{\emptyset\}$

(d) $A \cap \{\emptyset\}$

(e) $\{A, A\}$

(f) $2^A \cup A$

(g) $2^A \cup \{A\}$

Exercise 3. What's wrong with the following induction proof?

We prove that given n distinct lines in the plane, no two of them parallel, all these lines pass through a common point. The proof proceeds by induction. When $n = 1$ there is only one line and the claim is clearly true. Suppose it is true for $n = k$, and consider some set of $k+1$ lines, denoted $\ell_1, \ell_2, \ldots, \ell_{k+1}$. By the induction hypothesis, the lines $\ell_1, \ell_2, \ldots, \ell_k$ all pass through a common point x, and the lines $\ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_{k+1}$ all pass through a common point y. The lines ℓ_1 and ℓ_{k-1} belong to both sets, so x and y lie on both ℓ_1 and ℓ_{k-1}. However, ℓ_1 and ℓ_{k-1} intersect at exactly one point, so $x = y$ and all the lines pass through a common point. This completes the proof.

Exercise 4. Prove by induction:

(a)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

(b) i. Assuming $r \neq 1$,

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

ii. After proving this by induction, derive the same result by setting $S = \sum_{i=0}^{n} r^i$, multiplying this equation by r, and solving the two equations for S.

iii. Conclude that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Proof tip: It is often useful to "unpack" the $\sum$'s, that is, to write out the summation in your drafts with "...", just for yourself, to get a visual idea of what the summation "looks like".

Exercise 5. The Fibonacci sequence is defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \qquad \text{for all } n \geq 3. \end{aligned}$$

Prove that

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Exercise 6. In this exercise we will explore some of the properties of $\sum$ and $\prod$.

(a) Suppose $f(i)$ is some function of i, and $n, m \in \mathbb{N}^+$. Prove that

i.
$$\sum_{i=1}^{n} af(i) = a\sum_{i=1}^{n} f(i).$$

ii.
$$\prod_{i=1}^{n} af(i) = a^n \prod_{i=1}^{n} f(i).$$

(b) Simplify the following expressions and give a short justification of your solution:

i.
$$\sum_{i=1}^{n} a.$$

ii.
$$\prod_{j=1}^{n} f(i).$$

(c) Suppose $f(i,j)$ is some function of i and j. Is it true that

i.
$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(i,j) = \sum_{j=1}^{m}\sum_{i=1}^{n} f(i,j) \ ?$$

ii.
$$\prod_{i=1}^{n}\prod_{j=1}^{m} f(i,j) = \prod_{j=1}^{m}\prod_{i=1}^{n} f(i,j) \ ?$$

iii.
$$\prod_{i=1}^{n}\sum_{j=1}^{m} f(i,j) = \sum_{j=1}^{m}\prod_{i=1}^{n} f(i,j) \ ?$$

Prove *one* of the formulas that you believe to be true. For *one* of the others, give and substantiate a counterexample.

Exercise 7. Extra credit:

(a) Let us draw n lines in the plane in such a way that no two are parallel and no three intersect in a common point. Prove that the plane is divided into exactly

$$\frac{n(n+1)}{2} + 1$$

parts by the lines.

(b) Similarly, consider n planes in the 3-dimensional space in general position. (No two are parallel, any three have exactly one point in common, and no four have a common point.) What is the number of regions into which these planes partition the space?