CS 103X: Discrete Structures
Homework Assignment 4

Due February 10, 2006

Exercise 1. Compute the following without using computer software. You should find Fermat's Little Theorem useful for some of these.

(a) The last decimal digit of 3^{1000}.

(b) 3^{1000} rem 31.

(c) $3/16$ in $\mathbb{Z}_{31}$.

Exercise 2. Prove or disprove:

(a) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(b) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

(c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Exercise 3. Examples of relations:

(a) Find relations R and S on some set A, such that $R \circ S \neq S \circ R$.

(b) Find a relation R on a finite set A, such that $R^n \neq R^{n+1}$ for every $n \in \mathbb{N}^+$.

Exercise 4. Give an example of a function $f : \mathbb{N} \to \mathbb{Z}$ that is:

(a) Neither injective nor surjective.

(b) Injective but not surjective.

(c) Surjective but not injective.

(d) Surjective and injective.

Exercise 5. Let R and S be equivalences on a set A. Decide which of the following are necessarily also equivalences on A; prove or give a counterexample. Then assume that R and S are partial orders on A and decide which of the following are necessarily partial orders on A; again, prove or give a counterexample.

(a) $R \cap S$

(b) $R \cup S$

(c) $R \setminus S$

(d) $R \circ S$

Exercise 6. Extra credit: Let R be a relation on a set A, and T be the transitive closure of R. Prove:

(a) T is transitive.

(b) T is the smallest transitive relation that contains R. (That is, if U is a transitive relation on A and $R \subseteq U$, then $T \subseteq U$.)

(c) If $|A| = n$ then

$$T = \bigcup_{i=1}^{n} R^i.$$