CS 103X: Discrete Structures
Homework Assignment 9

Due March 17, 2006

Exercise 1 (20 points)**.** Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v. The *diameter* $\mathrm{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G. (That is, $\max_{u,v \in V} d_G(u, v)$.) The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v of G is defined as $\max_{u \in V} d_G(u, v)$. Finally, the *radius* $\mathrm{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G, namely $\min_{v \in V} \mathrm{ecc}(v)$. Prove:

(a) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

(b) For every $n \in \mathbb{N}^+$, there are connected graphs G_1 and G_2 with $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$ and $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$.

Exercise 2 (20 points)**.** Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Prove that if all vertex degrees in a graph G are even then G has no bridge.

Exercise 3 (20 points)**.** Prove that given a connected graph $G = (V, E)$, the degrees of all vertices of G are even if and only if there is a set of edge-disjoint cycles in G that cover the edges of G. (That is, the edge set of G is the disjoint union of the edge sets of these cycles.)

Exercise 4 (20 points)**.** For any $k \in \mathbb{N}^+$, prove that a k-regular bipartite graph has a perfect matching.

Exercise 5 (20 points)**.** Let G be a simple graph with n vertices and k connected components.

(a) What is the minimum possible number of edges of G?

(b) What is the maximum possible number of edges of G?