CS 103X: Discrete Structures
Homework Assignment 2

Due January 25, 2007

Exercise 1 (10 points)**.** Show that the sum of the first n odd natural numbers is n^2.

Exercise 2 (10 points)**.** What's wrong with the following induction proof?

> We prove that for any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, $a^n = 1$. The proof proceeds by strong induction. For the induction basis, $a^0 = 1$ and the claim holds. Assume that the claim holds for all k up to n. Then
>
> $$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$
>
> This proves the claim.

Exercise 3 (10 points)**.** Prove by induction that, for any set A, $|2^A| = 2^{|A|}$.

Exercise 4 (10 points)**.** Prove Bernoulli's inequality: For any $n \in \mathbb{N}$ and $r \in \mathbb{R}$, such that $r > -1$,

$$(1 + r)^n \geq 1 + rn.$$

Exercise 5 (10 points)**.** Prove the strong induction principle from the principle of induction. Conclude that the two principles are equivalent. (That is, anything that can be derived from one, can also be derived from the other.)

Exercise 6 (10 points)**.** Suppose $f(i, j)$ is a function of i and j, and $n \in \mathbb{N}^+$. Prove or give a counterexample:

$$\sum_{i=1}^{n} \sum_{j=1}^{i} f(i, j) \;=\; \sum_{j=1}^{n} \sum_{i=j}^{n} f(i, j)$$

If the sums are replaced with products, does your conclusion change?

Exercise 7 (20 points)**.** Many roots are irrational:

(a) Prove that $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$ are irrational. (Hint: For $\sqrt{3}$, use the fact that every integer is of the form $3n$, $3n + 1$, or $3n + 2$.) Why doesn't the same proof technique imply that $\sqrt{4}$ is irrational?

(b) Prove that $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} + \sqrt{6}$ are irrational.

Exercise 8 (20 points)**.** Consider n lines in the plane so that no two are parallel and no three intersect in a common point. What is the number of regions into which these lines partition the plane? Prove.

For example, the lines in the following diagram partition the plane into seven regions: