CS 103X: Discrete Structures
Homework Assignment 4

Due February 15, 2007

Exercise 1 (20 points)**.** For each of the following relations, state whether they fulfill each of the 4 main properties - reflexive, symmetric, antisymmetric, transitive. Briefly substantiate each of your answers.

(a) The coprime relation on $\mathbb{Z}$. (Recall that $a, b \in \mathbb{Z}$ are coprime if and only if $\gcd(a, b) = 1$.)

(b) Divisibility on $\mathbb{Z}$.

(c) The relation T on $\mathbb{R}$ such that aTb if and only if $ab \in \mathbb{Q}$.

Exercise 2 (20 points)**.** Prove that each of the following relations $\sim$ is an equivalence relation:

(a) For positive integers a and b, $a \sim b$ if and only if a and b have exactly the same prime factors, up to repetitions. (For example, $6 = 2 \times 3$ and $432 = 2^4 \times 3^3$ are related by $\sim$, but $18 = 2 \times 3^2$ and $10 = 2 \times 5$ are not.)

(b) For integers a and b, $a \sim b$ if and only if $a + 3b$ is divisible by 4.

(c) A sequence of real numbers $x_1, x_2, x_3 \ldots$ has a *limit* L if for any real number $\varepsilon > 0$, there is some integer n such that $|x_i - L| < \varepsilon$ for all $i > n$. (**Warning:** The condition in the above definition must hold for *all* possible $\varepsilon > 0$, not just one value of ε. For each ε there should be a corresponding n.) Let $A = a_1, a_2, a_3, \ldots$ and $B = b_1, b_2, b_3, \ldots$ be two sequences of real numbers. Then $A \sim B$ if and only if the sequence $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ has the limit 0.

(d) Let S be some set and T be a subset of S. For subsets A and B of S, say $A \sim B$ if and only if $(A \cup B) \setminus (A \cap B) \subseteq T$.

Exercise 3 (20 points)**.** Let A be a set. Given a relation R on A, define a relation S by $xSy \Leftrightarrow (xRy \text{ and } yRx)$, and a relation T by $xTy \Leftrightarrow (xRy \text{ and } y\not{R}x)$.

(a) Show that S is symmetric and T antisymmetric.

(b) Prove that $xRy \Leftrightarrow (xSy \text{ or } xTy)$.

(c) Show that if R is transitive, then S and T are also transitive, but that the reverse does not hold.

Exercise 4 (20 points)**.** Powers of relations:

(a) Prove that if R is a relation on a finite set A, there exist $n, m \in \mathbb{N}^+$, such that $R^n = R^m$.

(b) Prove that the claim in (a) need not hold if the set A is infinite.

Exercise 5 (20 points)**.** For each of the following pairs of sets, define a bijection between the two. You can choose which set is the domain and which is the codomain. You should state a precise rule that maps each member of the domain to a member of the codomain. (A little drawing is not a precise rule.) Provide a brief justification why your function is a bijection, but there is no need for a formal proof.

(a) $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$.

(b) $\mathbb{N}$ and $\mathbb{Z}$.

(c) $\mathbb{N}$ and F, where $F = \{a \in \mathbb{Z} : a \equiv_5 0\}$.

(d) $\mathbb{N}^+$ and $\mathbb{Q}^+$, where $\mathbb{Q}^+ = \{\frac{a}{b} : a, b \in \mathbb{N}^+\}$. (For the purposes of this question, two elements a/b and c/d in $\mathbb{Q}^+$ are considered the same only if $a = c$ and $b = d$. Thus 2/3 and 4/6 are regarded as distinct.)

For general education: An infinite set is said to be *countable* if it has the same cardinality as $\mathbb{N}$. The solution to the last question above can be easily extended to show that $\mathbb{Q}$ is countable. The set $\mathbb{R}$, on the other hand, is not countable.