CS 103X: Discrete Structures
Homework Assignment 6

Due March 1, 2007

Exercise 1 (15 points)**.** Warm-up:

(a) We have n married couples who are to sit at a round table of $2n$ spots. How many arrangements are possible if all $2n$ rotations of a given arrangement are considered equivalent and each person sits next to his/her spouse?

(b) A cop goes into a donut store and wishes to get a dozen. How many options does the officer have if s/he can choose from 5 different types of donuts and wishes to get at least one of each?

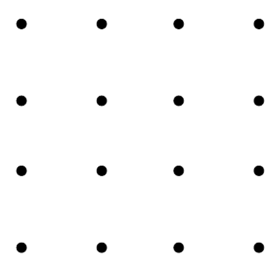
(c) The grocer sells six types of apples. You want to buy a bag of five, with no more than two from each type. How many options do you have?

Exercise 2 (10 points)**.** Consider the sequence of the first $2n$ positive integers. In how many ways can you order it such that no two consecutive terms have a sum divisible by 2?

Exercise 3 (10 points)**.** A Silicon Valley question: How many possible six-figure salaries (in whole dollar amounts) are there that contain some digit at least twice? (Hint: How about ones that do not contain any digit more than once?)

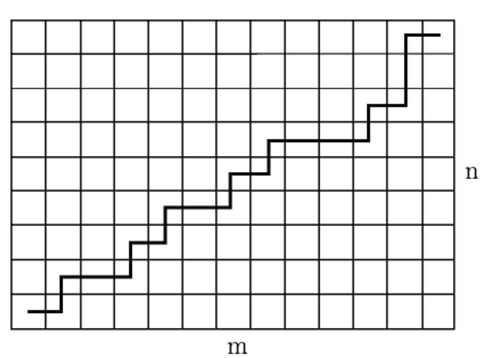
Exercise 4 (10 points)**.** A company board with n members sits down at a circular meeting table with n seats. Everybody knows that the chairman will go ballistic if seated in the chair closest to the window. How many safe seating arrangements are there?

Exercise 5 (10 points)**.** Consider a regular 4×4 grid of sixteen points, as in this picture:



How many triangles can be formed whose corners lie on the grid? A triangle has to have nonzero area.

Exercise 6 (10 points)**.** After King Arthur's knights got bored trying out all the possible ways they can sit at the round table, they discovered a more exciting pursuit. The King's courtyard was tiled with square tiles and when viewed from above looked like a perfect $m \times n$ grid. The knights sought in vain to answer the following question: Suppose Lancelot starts from the southwesternmost tile and repeatedly steps to the north or to the east, advancing one tile at a time, until he gets to the opposite (northeastern) end of the courtyard. How many such walks are there? Here is an example:



Can you help the noble, but, alas, combinatorially inept, knights?

Exercise 7 (15 points)**.** A phone number is a 7-digit sequence that does not start with 0.

(a) Call a phone number *lucky* if its digits are in nondecreasing order. For example, 1112234 is lucky, but 1112232 is not. How many lucky phone numbers are there? (10 points)

(b) A phone number is *very lucky* if its digits are strictly increasing, such as with 1235689. How many very lucky phone numbers are there? (5 points)

Exercise 8 (20 points)**.** Consider the expression $(ax + by)^n$.

(a) Given $a = 4$ and $b = 5$, find an n such that the expansion of $(ax + by)^n$ has consecutive terms with the same coefficients, namely terms $c_1 x^p y^q$ and $c_2 x^{p-1} y^{q+1}$ with $c_1 = c_2$. Include those two terms in your answer.

(b) Prove that it is impossible to have three consecutive terms (defined as in part (a)) with the same coefficients regardless of the values of $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.