CS 103X: Discrete Structures
Homework Assignment 7

Due March 8, 2007

Exercise 1 (10 points)**.** In a survey on the gelato preferences of college students, the following data was obtained:

- 78 like mixed berry
- 32 like irish cream
- 57 like tiramisu
- 13 like both mixed berry and irish cream
- 21 like both irish cream and tiramisu
- 16 like both tiramisu and mixed berry
- 5 like all three flavours above
- 14 like none of these three flavours

How many students were surveyed?

Exercise 2 (10 points)**.** In a mathematics contest with three problems, 80% of the participants solved the first problem, 75% solved the second and 70% solved the third. Prove that at least 25% of the participants solved all three problems. (The claim might seem obvious — find a *proof*.)

Exercise 3 (20 points)**.**

(a) What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 = 50,$$

such that $x_i \geq 0$ for each $1 \leq i \leq 3$?

(b) What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 = 50,$$

such that $0 \leq x_i \leq 9$ for each $1 \leq i \leq 3$?

Exercise 4 (10 points)**.** Let $p(1), p(2), \ldots, p(n)$ be some permutation of the first n positive integers, where n is odd. Prove that the product

$$\prod_{i=1}^{n}(i - p(i))$$

is necessarily even. (Assume as usual that an even number need not be positive.) Is the condition that n is odd necessary?

Exercise 5 (15 points)**.** Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them. Prove that there are two numbers i, j in this sample such that $i|j$.

Exercise 6 (15 points)**.** For each of the following pairs of functions $f, g : \mathbb{N}^+ \to \mathbb{R}$, state with a brief justification whether $f(x)$ is $O(g(x))$, $\Omega(g(x))$, $\Theta(g(x))$, or none of the above.

(a) $f(x) = x^{x^2}$, $g(x) = 2^{2^x}$

(b) $f(x) = \cos(x)$, $g(x) = 2^x \sin(x)$ (Note: x measured in degrees for the trigonometric functions)

(c) $f(x) = \sqrt[x]{x}$, $g(x) = \log_x x$

Exercise 7 (20 points)**.** Prove or disprove the following properties:

(a) For $f, g, p, q : \mathbb{N}^+ \to \mathbb{R}$: If $f(n) = O(p(n))$ and $g(n) = O(q(n))$, then $f(g(n)) = O(p(q(n)))$.

(b) For $f, p : \mathbb{N}^+ \to \mathbb{R}$ and $g, q : \mathbb{N}^+ \to \mathbb{N}^+$:If $f(n) = O(p(n))$, $g(n) = O(q(n))$, and $p(n), q(n) > 0$ for all n, then $(f(n))^{g(n)} = O((p(n))^{q(n)})$.