CS 103X: Discrete Structures
Homework Assignment 8

Due March 15, 2007

Exercise 1 (10 points)**.** The complement of a graph $G = (V, E)$ is the graph

$$\big(V,\ \{\{x, y\} : x, y \in E, x \neq y\} \setminus E\big).$$

A graph is *self-complementary* if it is isomorphic to its complement.

(a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such graphs.

(b) Find examples of self-complementary simple graphs with 4 and 5 vertices.

Exercise 2 (10 points)**.** Prove that if a graph has at most m vertices of degree at most n and all other vertices have degree at most k, with $k < n$ and $m < n$, then the graph is colorable with $m + k + 1$ colors.

Exercise 3 (30 points)**.** Prove or disprove, for a graph G on a finite set of n vertices:

(a) If every vertex of G has degree 2, then G contains a cycle.

(b) If G is disconnected, then its complement is connected.

(c) If T is a non-cyclic tour in G, and no strictly longer tour in G contains T, then both endpoints of T have odd degree.

Exercise 4 (15 points)**.** Consider m graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, …, $G_m = (V_m, E_m)$. Their union can be defined as

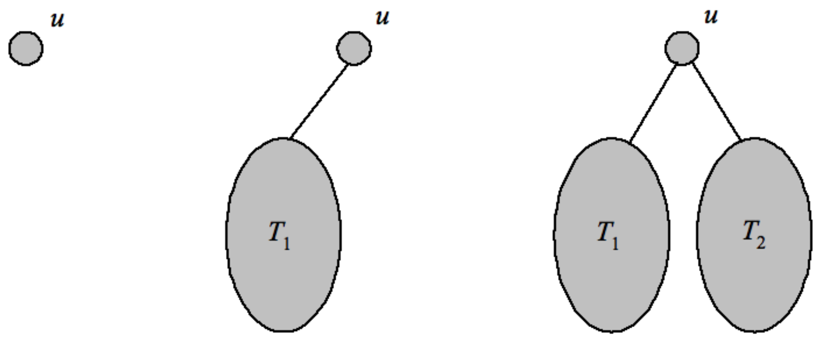
$$\bigcup_{i=1}^{m} G_i = \left(\bigcup_{i=1}^{m} V_i, \bigcup_{i=1}^{m} E_i\right).$$

Show that, for any natural number $n \geq 2$, the clique K_n can be expressed as the union of k bipartite graphs if $n \leq 2^k$.

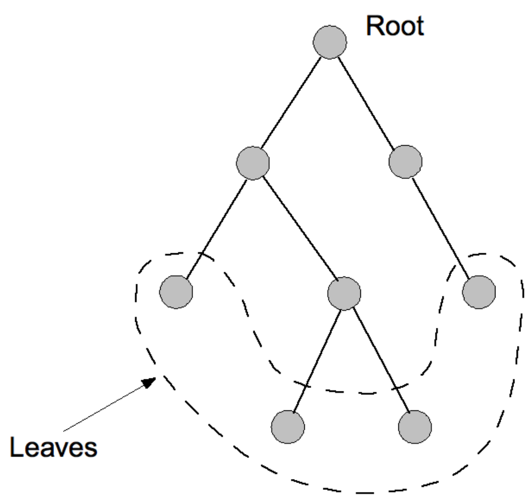
Exercise 5 (15 points)**.** A binary tree is defined inductively as follows:

- A single vertex u defines a binary tree with root u.
- A vertex u linked by edges to the roots of one or two binary trees defines a binary tree with root u.

The following figure illustrates the three possibilities:



T_1 and T_2 are called *subtrees*, u is the *parent* of the roots of the subtrees, and these roots are *children* of u. The vertices of a binary tree without any children are called *leaves*. Here's an example of a binary tree:



The *distance* between two vertices of a tree is the number of edges in the shortest path connecting them. The *height* of the tree is the maximum distance between the root and a leaf. Prove that the height of a binary tree with n vertices is at least $\log_2 n$. (Hint: Strong induction.)

Exercise 6 (20 points)**.** Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a bridge if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Let G be a bipartite k-regular graph for $k \geq 2$. Prove that G has no bridge.