CS 103X: Discrete Structures
Homework Assignment 6

Due March 1, 2007

Exercise 1 (15 points)**.** Warm-up:

(a) We have n married couples who are to sit at a round table of $2n$ spots. How many arrangements are possible if all $2n$ rotations of a given arrangement are considered equivalent and each person sits next to his/her spouse?

(b) A cop goes into a donut store and wishes to get a dozen. How many options does the officer have if s/he can choose from 5 different types of donuts and wishes to get at least one of each?

(c) The grocer sells six types of apples. You want to buy a bag of five, with no more than two from each type. How many options do you have?

Solution

(a) Consider this as a problem of n things at a table instead of $2n$, treating each couple as a single unit. From the lecture notes, there are $(n-1)!$ arrangements of couples. Then each couple can be arranged in one of two ways, so multiplying n times by 2 gives $2^n(n-1)!$.

(b) If the cop wants at least one of each type, he/she is really only choosing 7 donuts, not 12. Thus this is equivalent to choosing 7 unordered items from 5 types with repetition; the answer is $\binom{7+5-1}{7} = \binom{11}{7} = 330$.

(c) This is equivalent to the total number of ways to choose 5 apples from 6 types, minus the number of ways to choose 5 from 6 types with at least three of one kind. (Three of one kind and three of another kind are impossible, since the overall number of apples chosen is five.) The first number is straightforward unordered selection with replacement; it's $\binom{6+5-1}{5} = \binom{10}{5} = 252$. For the second part, first choose three of one type, there are 6 ways to do this. Then pick 2 of any type; there are $\binom{6+2-1}{2}$ ways. Thus the total is $6 \times \binom{7}{2} = 126$ ways, and subtracting them gives 126 possibilities.

Exercise 2 (10 points)**.** Consider the sequence of the first $2n$ positive integers. In how many ways can you order it such that no two consecutive terms have a sum divisible by 2?

Solution Two consecutive terms will sum to an even number if both are odd or both are even, so consecutive terms must always have opposite parities. Since there will be n numbers of each parity, the only way to arrange them is to have all odd-numbered terms be odd and all even-numbered terms even, or vice versa, so 2 possibilities. After that decision, you can have any arrangement of the n odd and n even numbers, so the total number of ways is $2(n!)^2$.

Exercise 3 (10 points)**.** A Silicon Valley question: How many possible six-figure salaries (in whole dollar amounts) are there that contain some digit at least twice? (Hint: How about ones that do not contain any digit more than once?)

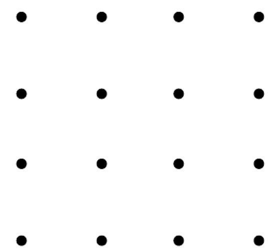
Solution The total number of possible six-figure salaries, without any restrictions, is 9×10^5 (digits $1 \ldots 9$ in the first place and digits $0 \ldots 9$ in the rest). Now let's find the number of salaries that contain no digit more than once. This is $9 \times (9)_5$, since there are 9 possibilities for the first digit (0 is not allowed) and for the remaining 5 we need to choose 5 digits out of 9 in order without repetition. Subtract the number of distinct-digit salaries from the overall number of possible salaries gives us the desired number of salaries that contain some digit at least twice. Specifically,

$$9 \times 10^5 - 9 \times (9)_5 = 763920.$$

Exercise 4 (10 points)**.** A company board with n members sits down at a circular meeting table with n seats. Everybody knows that the chairman will go ballistic if seated in the chair closest to the window. How many safe seating arrangements are there?

Solution The problem is simple once we make the initial observation that circular symmetry is broken by the presence of the window, and the situation is analogous to seating people in a row. (The chair C closest to the window, the chair immediately clockwise of C, the chair two steps clockwise of C, and so on.) The chairman may be seated in $n-1$ ways — the chair closest to the window is excluded. The remaining people may be arranged in $(n-1)!$ ways in the remaining $n-1$ chairs. The complete arrangements are all distinct, since even if each person has the same neighbors in two different arrangements, the window acts as a marker that yields an ordering on the chairs. So the total number of ways is $(n-1) \times (n-1)!$.

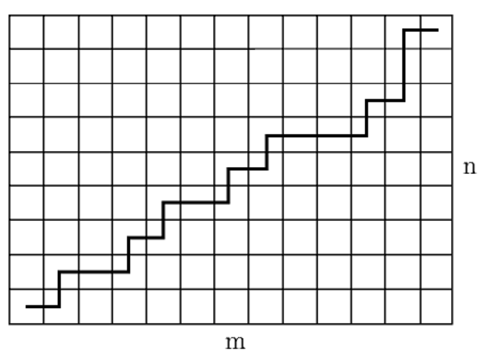
Exercise 5 (10 points)**.** Consider a regular 4×4 grid of sixteen points, as in this picture:



How many triangles can be formed whose corners lie on the grid? A triangle has to have nonzero area.

Solution The number of ways of picking three points from the grid is $\binom{16}{3} = 560$. These three points will form a degenerate triangle (one with zero area) if and only if they all lie in a straight line. This straight line may be one of the rows, one of the columns, or one of the diagonals. The number of ways of picking three points, all from one row of the grid, is $4 \times \binom{4}{3} = 16$ (pick the row, then pick 3 points within the row). Symmetrically, the number of ways of picking three points, all from the same column, is 16. Now let's try the diagonals. There are three "bottom-left to top-right" diagonals with at least three points — the middle one has 4 points and the others have 3 each. The number of ways of picking three points, all from one of these diagonals, is $\binom{4}{3} + 2 \times \binom{3}{3} = 6$. The situation for "bottom-right to top-left" diagonals is identical and also gives 6 possibilities. So the total number of ways of picking a degenerate triangle is $2 \times 16 + 2 \times 6 = 44$, and the number of ways of picking a non-degenerate triangle is $560 - 44 = 516$.

Exercise 6 (10 points)**.** After King Arthur's knights got bored trying out all the possible ways they can sit at the round table, they discovered a more exciting pursuit. The King's courtyard was tiled with square tiles and when viewed from above looked like a perfect $m \times n$ grid. The knights sought in vain to answer the following question: Suppose Lancelot starts from the southwesternmost tile and repeatedly steps to the north or to the east, advancing one tile at a time, until he gets to the opposite (northeastern) end of the courtyard. How many such walks are there? Here is an example:



Can you help the noble, but, alas, combinatorially inept, knights?

Solution In total Lancelot must take $(m-1)$ steps to the east and $(n-1)$ steps north. The number of distinct paths is the same as the number of ways to select the $(m-1)$ east steps and place them in the sequence of all $(m+n-2)$ steps, which is the same as picking $(m-1)$ members of the set $\{1, 2, ..., m+n-2\}$ without concern for order. Thus the answer is $\binom{m+n-2}{m-1}$.

Exercise 7 (15 points)**.** A phone number is a 7-digit sequence that does not start with 0.

(a) Call a phone number *lucky* if its digits are in nondecreasing order. For example, 1112234 is lucky, but 1112232 is not. How many lucky phone numbers are there? (10 points)

(b) A phone number is *very lucky* if its digits are strictly increasing, such as with 1235689. How many very lucky phone numbers are there? (5 points)

Solution

(a) A lucky phone number is uniquely identified by the choice of seven digits, with possible repetitions, since these digits must then be placed in sorted order. We can ignore the digit 0 altogether, since in a nondecreasing sequence 0 would have to occur in the first place and we explicitly rule out this possibility. The number of ways to pick an ordered sequence of seven non-zero digits with possible repetitions is thus $\binom{7+9-1}{7} = \binom{15}{7} = 6435$.

(b) As in the first part, a very lucky phone number is uniquely defined by the choice of seven distinct digits. We can again ignore the digit 0. The number of ways to pick seven distinct non-zero digits is $\binom{9}{7} = 36$, which is the number of very lucky phone numbers.

Exercise 8 (20 points)**.** Consider the expression $(ax + by)^n$.

(a) Given $a = 4$ and $b = 5$, find an n such that the expansion of $(ax + by)^n$ has consecutive terms with the same coefficients, namely terms $c_1 x^p y^q$ and $c_2 x^{p-1} y^{q+1}$ with $c_1 = c_2$. Include those two terms in your answer.

(b) Prove that it is impossible to have three consecutive terms (defined as in part (a)) with the same coefficients regardless of the values of $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

Solution

(a) From application of the binomial theorem, we know that $c_1 = \binom{n}{q} a^p b^q$ and $c_2 = \binom{n}{q+1} a^{p-1} b^{q+1}$. Equating these and dividing by $a^{p-1} b^q$ gives

$$\binom{n}{q} a = \binom{n}{q+1} b$$

Then expanding using the factorial form of $\binom{n}{q}$,

$$\frac{a(n!)}{q!p!} = \frac{b(n!)}{(q+1)!(p-1)!}$$

Dividing and multiplying out common terms of the factorials leaves us with

$$\frac{a}{p} = \frac{b}{(q+1)}$$

or

$$\frac{4}{p} = \frac{5}{(q+1)}$$

By inspection, an integer solution is $p = 4, q = 4$, so $n = 8$. The terms are then $\binom{8}{4}(4x)^4(5y)^4 = 11200000x^4y^4$ and $\binom{8}{5}(4x)^3(5y)^5 = 11200000x^3y^5$.

(b) We proceed by contradiction. Assuming that this is possible, we can follow the technique of part (a) using $c_0 = \binom{n}{q-1} a^{p+1} b^{q-1}$ and the same formulas for c_1 and c_2 as in (a) to find relations between a, b, n (remembering that $n = p + q$). Setting all of these equal to each other gives

$$\binom{n}{q-1} a^{p+1} b^{q-1} = \binom{n}{q} a^p b^q = \binom{n}{q+1} a^{p-1} b^{q+1}.$$

Now expand using factorials and divide out powers of a and b:

$$\frac{a^2(n!)}{(q-1)!(p+1)!} = \frac{ab(n!)}{q!p!} = \frac{b^2(n!)}{(q+1)!(p-1)!}.$$

Simplifying and splitting into two separate equations gives

$$\frac{a^2}{p(p+1)} = \frac{ab}{pq} \qquad \text{and} \qquad \frac{ab}{pq} = \frac{b^2}{q(q+1)}.$$

Each of these can be solved for a ratio of $\frac{a}{b}$: The left gives $\frac{a}{b} = \frac{p+1}{q}$ and the right gives $\frac{a}{b} = \frac{p}{q+1}$. Thus we can say $\frac{p+1}{q} = \frac{p}{q+1}$, so $pq = (p+1)(q+1) = pq + p + q + 1$, so $p + q + 1 = 0$. Since p, q are nonnegative integers, this has no solution; thus we have reached a contradiction and the proof is complete.