CS 103X: Discrete Structures
Homework Assignment 7 — Solutions

Exercise 1 (10 points)**.** In a survey on the gelato preferences of college students, the following data was obtained:

- 78 like mixed berry
- 32 like irish cream
- 57 like tiramisu
- 13 like both mixed berry and irish cream
- 21 like both irish cream and tiramisu
- 16 like both tiramisu and mixed berry
- 5 like all three flavours above
- 14 like none of these three flavours

How many students were surveyed?

Solution Let the set of students who like mixed berry be M, those who like tiramisu be T and those who like irish cream be I. Then, by the inclusion-exclusion principle, the number of students who like at least one of the flavours is

$$\begin{aligned} |M \cup T \cup I| &= |M| + |T| + |I| - |M \cap T| - |T \cap I| - |M \cap I| + |M \cap T \cap I| \\ &= 78 + 32 + 57 - 16 - 21 - 13 + 5 \\ &= 122 \end{aligned}$$

Now there are an additional 14 who like none of these three flavours, so the total number of students surveyed was $122 + 14 = 136$.

Exercise 2 (10 points)**.** In a mathematics contest with three problems, 80% of the participants solved the first problem, 75% solved the second and 70% solved the third. Prove that at least 25% of the participants solved all three problems. (The claim might seem obvious — find a *proof*.)

Solution Let the total number of participants be $n > 0$ (if $n = 0$, the proof is trivial). Denote the set of people who missed the first problem by A, the set of people who missed the second by B, and the set who missed the third by C. We know that $|A| = n - 0.8n = 0.2n$, $|B| = n - 0.75n = 0.25n$ and $|C| = n - 0.7n = 0.3n$. We also know, from the lecture notes, that

$$|A \cup B \cup C| \leq |A| + |B| + |C| = 0.2n + 0.25n + 0.3n = 0.75n$$

The set of people who solved all three problems is the complement of $A \cup B \cup C$ (the set who missed at least one problem), so it has size

$$n - |A \cup B \cup C| \geq n - 0.75n = 0.25n$$

Therefore at least 25% of the participants solved all three problems.

Exercise 3 (20 points).

(a) What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 = 50,$$

such that $x_i \geq 0$ for each $1 \leq i \leq 3$?

(b) What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 = 50,$$

such that $0 \leq x_i \leq 19$ for each $1 \leq i \leq 3$?

Solution

(a) We need to distribute 50 "units" between 3 variables. The situation is equivalent to distributing 50 balls into 3 bins, or analogously, selecting an unordered collection of 50 balls of 3 types, where those of the first type will go in the first bin, those of the second type in the second bin, and those of the third type in the third bin. The number of options for this is $\binom{3+50-1}{50} = \binom{52}{50} = 1326$. This is also the requested number of solutions.

(b) For $1 \leq i \leq 3$, let S_i be the set of solutions in which $x_i \geq 20$. Then the requested number of solutions is

$$1326 - \left|\bigcup_{i=1}^{3} S_i\right|,$$

using the result of part (a). For $1 \leq i < j \leq 3$, define $S_{i,j} = S_i \cap S_j$. Finally, let $S_{1,2,3} = S_1 \cap S_2 \cap S_3$. The inclusion exclusion principle implies that

$$\left|\bigcup_{i=1}^{3} S_i\right| = \sum_{i=1}^{3} |S_i| - |S_{1,2}| - |S_{2,3}| - |S_{1,3}| + |S_{1,2,3}|.$$

Let us analyze the summands on the right side in turn. $|S_1|$ is the number of solutions in which $x_1 \geq 20$. This is equivalent to counting solutions to the equation

$$x_1' + x_2 + x_3 = 30,$$

where $x_1' = x_1 - 20$, and thus $x_1', x_2, x_3 \geq 0$. The argument in part (a) implies that $|S_1| = \binom{32}{2} = 496$. The derivation for $|S_2|$ and $|S_3|$ is symmetric.

Similarly, $|S_{1,2}|$ is equal to the number of nonnegative solutions to

$$x_1' + x_2' + x_3 = 10,$$

which is 66 and is also equal to $|S_{2,3}|$ and $|S_{1,3}|$. Finally, $|S_{1,2,3}| = 0$. Putting all the values together we get

$$\left|\bigcup_{i=1}^{3} S_i\right| = 3 \times 496 - 3 \times 66 = 1290$$

and the requested number of solutions is

$$1326 - 1290 = 36.$$

Exercise 4 (10 points). Let $p(1), p(2), \ldots, p(n)$ be some permutation of the first n positive integers, where n is odd. Prove that the product

$$\prod_{i=1}^{n} (i - p(i))$$

is necessarily even. (Assume as usual that an even number need not be positive.) Is the condition that n is odd necessary?

Solution First, observe that if any one of the factors in a product of integers is even, the whole product must be even. Assume, for the sake of contradiction, that all the factors $(i - p(i))$ are odd. This implies that for every pair $\{i, p(i)\}$, either i is odd and $p(i)$ is even, or $p(i)$ is odd and i is even. Now since n is odd, i is odd in exactly $(n+1)/2$ cases, and in each of those cases $p(i)$ must be even. But there are only $(n-1)/2$ even numbers between 1 and n, so by the pigeonhole principle two of the $p(i)$'s must be equal. This is a contradiction since p is a permutation of distinct numbers.

The condition that n is odd *is* necessary. Else, for even n, consider this permutation:

$$p(i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even} \end{cases}$$

It is easy to verify that this is indeed a permutation (it would not be if n was odd since it would map n to $n+1$), and that it swaps successive pairs of elements. Evidently, each of the factors $(i - p(i))$ has absolute value 1, and hence the whole product has absolute value 1 and is odd.

Exercise 5 (15 points)**.** Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them. Prove that there are two numbers i, j in this sample such that $i|j$.

Solution For $1 \leq i \leq n$, define a subset A_i of $\{1, 2, \ldots, 2n\}$ as follows:

$$A_i = \{x \in A : \exists a \in \mathbb{N} : x = (2i-1) \times 2^a\}.$$

The set A_i includes the i-th odd number and the products of that number with powers of 2, up to $2n$. These sets have the property that for any two distinct numbers $a, b \in A_i$, either $a \mid b$ or $b \mid a$. Crucially, every integer between 1 and $2n$ belongs to exactly one such subset, because we can uniquely represent *every* non-zero integer as the product of an odd integer and a (possibly zero-th) power of 2 (prove this by induction!). Thus we treat these n subsets as pigeonholes, and the $n+1$ numbers in the sample as pigeons. By the pigeonhole principle, at least one set A_i has two members of the sample and this produces the i, j to satisfy the claim.

Note that unlike many pigeonhole problems, the sets A_i have different cardinalities. For example A_1 has $\lfloor \log_2(2n) \rfloor + 1$ members, while $|A_i| = 1$ for $\lceil n/2 \rceil \leq i \leq n$. Also note that the property that every integer in $\{1, 2, \ldots, 2n\}$ is in *exactly* one A_i is not necessary for the proof: the pigeonhole principle works even if the "pigeonholes" overlap — do you see why?

Exercise 6 (15 points)**.** For each of the following pairs of functions $f, g : \mathbb{N}^+ \to \mathbb{R}$, state with a brief justification whether $f(x)$ is $O(g(x))$, $\Omega(g(x))$, $\Theta(g(x))$, or none of the above.

(a) $f(x) = x^{x^2}$, $g(x) = 2^{2^x}$

(b) $f(x) = \cos(x)$, $g(x) = 2^x \sin(x)$. (x is measured in degrees here.)

(c) $f(x) = \sqrt[x]{x}$, $g(x) = \log_x x$

Solution

(a) $f(x) = O(g(x))$. Taking the logarithm (base 2) of both functions gives $\log_2 f(x) = x^2 \log_2 x$ and $\log_2 g(x) = 2^x$. From the basic rules in the lecture notes about logarithms, polynomials, and exponents we know $x^2 \log_2 x = O(2^x)$. Thus $f(x) = O(g(x))$ (take the constant c from $x^2 \log_2 x = O(2^x)$ and set $c_1 = 2^c$ to get the constant for f, g; the n_0 constant is the same for both the logs of the functions and the originals).

(b) None of the above. Since x is in degrees for the trigonometric functions, when x is a multiple of 180, $g(x) = 0$ and $|f(x)| = 1$ and when $x \equiv_{180} 90$, $f(x) = 0$ and $|g(x)| > 0$. Thus no c exists with $|f(x)| \leq c|g(x)|$ or vice versa, so none of the relations apply.

(c) $f(x) = \Theta(g(x))$. Note that $g(x) = 1$ for all $x \in \mathbb{N}^+$. We can show that $f(x) \leq 2$ for all $x \in \mathbb{N}^+$: Consider the equation $y = f(x)$ and rearrange to $y^x = x$. Since $2^x \geq x$, $y \leq 2$ for all $x \in \mathbb{N}^+$. Clearly $f(x) > 1$ for all $x \in \mathbb{N}^+$. These bounds ensure that $g(x) < f(x) \leq 2g(x)$, so $f(x) = \Theta(g(x))$.

Exercise 7 (20 points)**.** Prove or disprove the following properties:

(a) For $f, g, p, q : \mathbb{N}^+ \to \mathbb{R}$, if $f(n) = O(p(n))$ and $g(n) = O(q(n))$, then $f(g(n)) = O(p(q(n))$.

(b) For $f, p : \mathbb{N}^+ \to \mathbb{R}$ and $g, q : \mathbb{N}^+ \to \mathbb{N}^+$, if $f(n) = O(p(n))$, $g(n) = O(q(n))$, and $p(n), q(n) > 0$ for all n, then $(f(n))^{g(n)} = O((p(n))^{q(n)})$.

Solution

(a) This does not hold for all functions f, g, p, q. A counterexample is $f(n) = \frac{1}{n^3}$, $g(n) = n$, $p(n) = \frac{1}{n^2}$, and $q(n) = n^2$. Since a smaller polynomial power is always big-O of a larger one, $f(n) = O(p(n))$ and $g(n) = O(q(n))$. The composed functions are $f(g(n)) = \frac{1}{n^3}$ and $p(q(n)) = \frac{1}{n^4}$. For any positive constant c, $f(g(n)) > c(p(q(n)))$ when $n > c$, so $f(g(n)) \neq O(p(q(n)))$. Note that observing that $p(q(n)) = O(f(g(n)))$ is not sufficient to conclude $f(g(n)) \neq O(p(q(n)))$.

(b) This also does not hold for all functions. For a simple example, take $f(n) = p(n) = 2$, $g(n) = 2n$, and $q(n) = n$. Clearly these fit the given big-O conditions. The combined functions are $(f(n))^{g(n)} = 2^{2n} = 4^n$ and $(p(n))^{q(n)} = 2^n$. $4^n \neq O(2^n)$ — for any positive constant c, when $n > \log_2 c$, $4^n > c \cdot 2^n$.