CS 103X: Discrete Structures
Homework Assignment 8 — Solutions

Exercise 1 (10 points)**.** The complement of a graph $G = (V, E)$ is the graph

$$(V, \{\{x, y\} : x, y \in V, x \neq y\} \setminus E).$$

A graph is *self-complementary* if it is isomorphic to its complement.

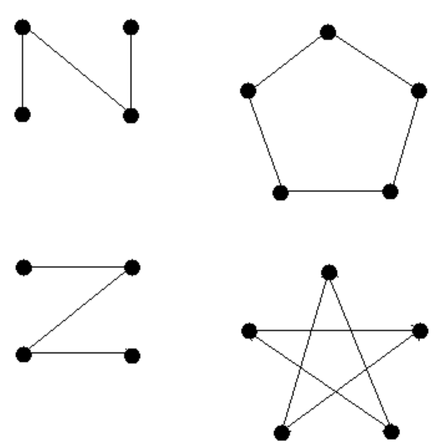
(a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such simple graphs.

(b) Find examples of self-complementary simple graphs with 4 and 5 vertices.

Solution

(a) Obviously, two isomorphic graphs must have the same number of edges. Thus for a graph with n vertices to be self-complementary, the total number of *possible* edges, $\binom{n}{2}$, must be even so that the graph and its complement can have the same number of edges. $\binom{2}{2} = 1$ and $\binom{3}{2} = 3$, so no graph with 2 or 3 vertices can be self-complementary.

(b) Here are the examples (top row) with their complements (bottom row). Convince yourself (with a bit of wire if necessary) that the pentagon and the five-pointed star are indeed isomorphic.



Exercise 2 (10 points)**.** Prove that if a graph has at most m vertices of degree at most n and all other vertices have degree of at most k, with $k < n$ and $m < n$, then the graph is colorable with $m + k + 1$ colors.

Solution First consider the reduced problem of coloring the graph minus the m vertices of degree at most n and all edges involving those vertices. From the lecture notes, since all remaining vertices have degree k or less, $k+1$ colors are enough for this reduced graph. Then if we restore the original graph and assign one color not used to each of the m vertices, the resulting graph will be colored using $m+k+1$ colors.

Exercise 3 (30 points)**.** Prove or disprove, for a graph G on a finite set of n vertices:

(a) If every vertex of G has degree 2, then G contains a cycle.

(b) If G is disconnected, then its complement is connected.

(c) If T is a non-cyclic tour in G, and no strictly longer tour in G contains T, then both endpoints of T have odd degree.

Solution

(a) True. Assume, for contradiction, that G has no cycle, and consider the longest path P in G (one must exist, since the graph is finite). Let v be the final vertex in P — since v has degree 2, it must have two edges e_1 and e_2 incident on it, of which one, say e_1, is the last edge of the path P. Then e_2 cannot be incident on any other vertex of P since that would create a cycle $(v, e_2, [\text{section of P ending in } e_1], v)$. So e_2 and its other endpoint are not part of P, and can be appended to P to give a strictly longer path, which contradicts our choice of P. Hence G must contain a cycle.

(b) True. Let G' denote the complement of G. Consider any two vertices u, v in G. If u and v are in different connected components in G, then no edge of G connects them, so there will be an edge $\{u, v\}$ in G'. If u and v are in the same connected component in G, then consider any vertex w in a *different* connected component (since G is disconnected, there must be at least one other connected component). By our first argument, the edges $\{u, w\}$ and $\{v, w\}$ exist in G', so u and v are connected by the path (u, w, v). Hence any two vertices are connected in G', so the whole graph is connected.

(c) True. This is similar to Theorem 15.1.1 for Eulerian tours. At each vertex of the tour T, label the incident edges as "entering" or "leaving" as in the lecture notes. At each endpoint of the tour, there is one leaving edge for every entering edge, except for exactly one edge (since the tour is non-cyclic) which is either leaving (if the vertex is the first one of the tour) or entering (if the vertex is the last one). This means an odd number of edges of the tour are incident at each endpoint. Assume, for contradiction, that an endpoint has even degree — since the tour contributes an odd number of edges to this count, there must be at least one non-tour edge incident at this vertex. This edge can be added to T to give a strictly longer tour that contains T, which is a contradiction. Hence both endpoints of T have odd degree.

Exercise 4 (15 points)**.** Consider m graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $\ldots$, $G_m = (V_m, E_m)$. Their union can be defined as

$$\bigcup_{i=1}^{m} G_i = \left(\bigcup_{i=1}^{m} V_i, \bigcup_{i=1}^{m} E_i \right).$$

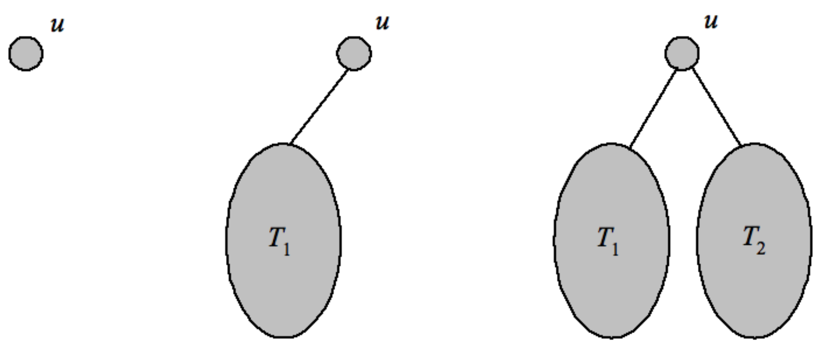
Show that, for any natural number $n \geq 2$, the clique K_n can be expressed as the union of k bipartite graphs if $n \leq 2^k$.

Solution We proceed by induction on k. For $k = 1$, there are two cliques: K_1 is just a single point, which is trivially a bipartite graph. K_2 is also a single bipartite graph (each vertex in its own group). Now, for the inductive step assume the claim holds for $k = m$. Now for any $n \leq 2^{m+1}$, let $a = \lfloor \frac{n}{2} \rfloor$ and $b = \lceil \frac{n}{2} \rceil$, and divide the vertices of K_n into disjoint sets A, B with $|A| = a$ and $|B| = b$. We can define a bipartite graph G with vertices $A \cup B$ and edges v, w with $v \in A, w \in B$. Removing all of these edges from K_n leave two cliques K_a and K_b. Since $a, b \leq 2^m$, each of these cliques can be represented as a union of m bipartite graphs. Since the two cliques have disjoint vertex sets, we can say that the union of a bipartite graph over the vertices of K_a and a bipartite graph over the vertices of K_b will still be a bipartite graph. Thus the two cliques together can be represented as the union of m bipartite graphs, and adding G to the union represents all of K_n as $m + 1$ bipartite graphs. This completes the inductive step and thus by induction the property holds for all k.

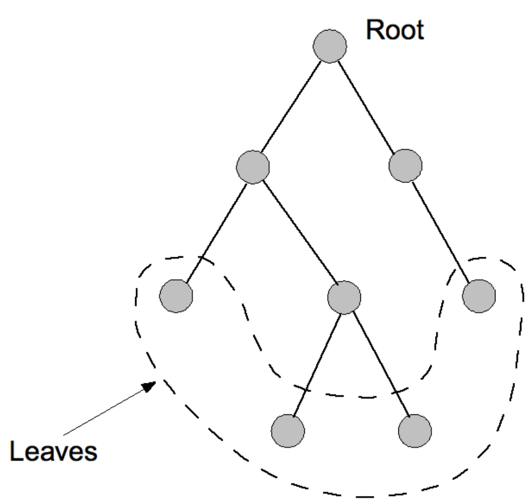
Exercise 5 (15 points)**.** A binary tree is defined inductively as follows:

- A single vertex u defines a binary tree with root u.
- A vertex u linked by edges to the roots of one or two binary trees defines a binary tree with root u.

The following figure illustrates the three possibilities:



T_1 and T_2 are called *subtrees*, u is the *parent* of the roots of the subtrees, and these roots are *children* of u. The vertices of a binary tree without any children are called *leaves*. Here's an example of a binary tree:



The *distance* between two vertices of a tree is the number of edges in the shortest path connecting them. The *height* of the tree is the maximum distance between the root and a leaf. Prove that the height of a binary tree with n vertices is at least $\lfloor \log_2 n \rfloor$. (Hint: use the strong induction principle.)

Solution The result follows from strong induction on n. The base case is the binary tree with one vertex, which is just the vertex itself. Obviously, this has height $0 = \lfloor \log_2 1 \rfloor$, so the base case holds. Now assume the result holds for all binary trees with at most m vertices, and consider a binary tree with $n = m + 1$ vertices. By definition, this can be viewed as a root vertex u plus two subtrees T_1 and T_2 — evidently T_1 and T_2 together contain m vertices. Assume that T_1 has p vertices — then T_2 has $m - p$ vertices. Since both p and $m - p$ are at most m, the induction hypothesis applies and the heights of the subtrees are at least $\lfloor \log_2 p \rfloor$ and $\lfloor \log_2(m - p) \rfloor$ respectively. The height h of the original tree is one more than the height of the "taller" subtree (because of the edge linking the root of the subtree to the root of the original tree), so we have

$$h = 1 + \max\{\lfloor \log_2 p \rfloor, \lfloor \log_2(m - p) \rfloor\}$$

Now at least one of the subtrees must have at least $\lceil \frac{m}{2} \rceil$ vertices, so we can write

$$h \geq 1 + \left\lfloor \log_2 \left\lceil \frac{m}{2} \right\rceil \right\rfloor$$

If m is odd, then this gives

$$\begin{aligned} h &\geq 1 + \left\lfloor \log_2 \frac{m+1}{2} \right\rfloor \\ &= 1 + \lfloor \log_2(m+1) - \log_2 2 \rfloor \\ &= 1 + \lfloor \log_2(m+1) - 1 \rfloor \\ &= 1 + \lfloor \log_2(m+1) \rfloor - 1 \\ &= \lfloor \log_2(m+1) \rfloor \end{aligned}$$

Also, if m is even, then we have

$$\begin{aligned} h &\geq 1 + \left\lfloor \log_2 \frac{m}{2} \right\rfloor \\ &\geq 1 + \lfloor \log_2 m - \log_2 2 \rfloor \\ &\geq 1 + \lfloor \log_2 m - 1 \rfloor \\ &\geq 1 + \lfloor \log_2 m \rfloor - 1 \\ &\geq \lfloor \log_2 m \rfloor \end{aligned}$$

But since m is even, $\lfloor \log_2 m \rfloor = \lfloor \log_2(m+1) \rfloor$ (the integer part of $\log_2 x$ increases by one only when x is a power of 2). So in either case, $h \geq \lfloor \log_2(m+1) \rfloor$, and the induction step holds. By the strong induction principle, this proves the result for all $n \in \mathbb{N}^+$.

Exercise 6 (20 points)**.** Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a bridge if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Let G be a bipartite k-regular graph for $k \geq 2$. Prove that G has no bridge.

Solution We will prove the result by contradiction. Assume G has a bridge $e = \{u, v\}$. Let's start with a couple of easy observations. Firstly, note that a bridge affects only the connected component it belongs to. Every connected component of a bipartite k-regular graph is itself bipartite k-regular, so we can assume, without loss of generality, that G is a *connected* bipartite k-regular graph. Secondly, removal of an edge can split a connected graph into at most two connected components

— to see why, observe that if we restore the edge, the graph should be connected, but three or more disjoint components cannot be linked by a single edge.

Now assume G has classes A and B, where $u \in A$ and $v \in B$. Removal of e splits G into disjoint components G_1 and G_2. Let A' be the set of vertices of A in G_1 and A'' be those in G_2 — both these sets are non-empty. Similarly let B', B'' be the vertices of B in G_1 and G_2 respectively. Observe that the bridge e must be the *only* edge linking G_1 and G_2, and assume without loss of generality that $u \in A'$ and $v \in B''$.

Now look at G_1, which is a bipartite graph with classes A' and B'. Since e is the only edge linking G_1 and G_2, every other edge of G incident on A' or B' is retained in G_1. So every vertex in A' and B' still has degree k in G_1, except u which has degree $k-1$. Let $a := |A'|$ and $b := |B'|$. Since no edge links two vertices in B' (bipartite property), the number of edges in G_1 is simply kb (every edge is incident to some vertex in B', so we can add up the degrees of the vertices in B'). Similarly, adding up the degrees in A' instead, the number of edges is $k(a-1)+k-1$. Equating the two formulæ, we have

$$\begin{array}{rrcl} & k(a-1)+k-1 & = & kb \\ \Rightarrow & k(a-b) & = & 1 \end{array}$$

But this implies $k = 1$, which contradicts the given condition that $k \geq 2$. Hence the bridge cannot exist.