CS 103X: Discrete Structures
Midterm

February 8, 2007

THE STANFORD UNIVERSITY HONOR CODE

- The Honor Code is an undertaking of the students, individually and collectively:
  - that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;
  - that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.
- The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid as far as practicable, academic procedures that create temptations to violate the Honor Code.
- While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to create optimal conditions for honorable academic work.

Exams are to be done individually and must represent original work—it is a violation of the honor code to copy or derive exam question solutions from other students or anyone at all, textbooks, or previous instances of this course.

I acknowledge and accept the honor code:

Signature: ______________________________

Name (print): ______________________________

EXAM RULES

- You have two hours to complete this exam.
- Do not include your scratch work with your exam. Please work the solutions out on another sheet of paper and then write your solutions neatly on the exam.
- You may use Prof. Koltun's lecture notes (Sections 1 through 6) and one double-sided cheat-sheet. You may *not* use any other material, such as your own course notes, homeworks, distributed homework solutions, other sets of lecture notes, books, computers, cell phones, crystal balls, Tarot cards, etc.
- Write clearly and neatly.
- Stagger your seats.

GOOD LUCK!

Exercise 1 (20 points)**.** Prove or disprove: For any three sets A, B, C,

(a) $C \setminus (A \cap B) = (C \setminus A) \cap (C \setminus B)$.

(b) $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$.

Exercise 2 (20 points)**.** You have been appointed Postmaster General for a new nation. Your job is to determine what denominations to issue stamps in. Unfortunately, the government wants to be able to charge an amount for each letter or package that corresponds exactly to its weight, so the postage fee could be any integral number of cents. The government is also cheap and wants to reduce costs by only printing a minimal number of types of stamps, and stamps of value less than 5 cents will not be allowed.

(a) What is the minimum number n of integer values $v_1, v_2, \ldots, v_n$ (where $v_i \geq 5$ for all i) such that all positive integers can be expressed as linear combinations of them? What is the precise condition that such a minimal set $v_1, v_2, \ldots, v_n$ has to satisfy? Prove. Does your answer apply to real life?

(b) Suppose the government caps the lowest possible package charge at c cents, for some $c > 50$. In a "real life" scenario, what is now the minimum number of stamp values $v_1, v_2, \ldots$ (where $v_i \geq 5$ for all i) that you can get away with so as to cover all package weights greater or equal to c? Prove.

Exercise 3 (20 points)**.** Given a rational number r and two irrational numbers a and b, prove or disprove:

(a) ab is irrational.

(b) ar is irrational.

(c) a^b is irrational.[1]

[1] Don't get stuck on this one.

Exercise 4 (20 points)**.** A perfect number is a natural number $n \geq 2$ with the property that the sum of all of n's divisors (including 1, but not n itself) is n. 6 and 28 are the first two examples. Prove that if $(2^p - 1)$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number. Using this property, find another perfect number besides 6 and 28.

Exercise 5 (20 points). Consider n planes in 3-dimensional space so that no two are parallel, any three have exactly one point in common, and no four have a common point. What is the number of 3-dimensional parts into which these planes partition the space? Prove. (You can use the 2-dimensional result without proof.)