

# CS103X: Discrete Structures

## Homework Assignment 1: Solutions

Due January 25, 2008

**Exercise 1** (10 Points). Prove or give a counterexample for each of the following:

- (a) If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .
- (b) If  $A \in B$  and  $B \in C$ , then  $A \in C$ .

**Solution:**

- (a) Consider any element  $a \in A$ . Since  $A \subseteq B$  every element of  $A$  is also an element of  $B$ , so  $a \in B$ . By same reasoning,  $a \in C$  since  $B \subseteq C$ . Thus every element of  $A$  is an element of  $C$ , so  $A \subseteq C$ .
- (b) Let  $A = \{1\}$ ,  $B = \{1, \{1\}\}$ , and  $C = \{\{1, \{1\}\}, 3, \{4\}\}$ . These sets satisfy  $A \in B$  and  $B \in C$ , but  $A \notin C$ .

**Exercise 2** (10 Points). If  $a(t)$ ,  $b(t)$ , and  $c(t)$  are the lengths of the three sides of a triangle  $t$  in *non-decreasing order* (i.e.  $a(t) \leq b(t) \leq c(t)$ ), we define the sets:

- $X := \{\text{Triangle } t : a(t) = b(t)\}$
- $Y := \{\text{Triangle } t : b(t) = c(t)\}$
- $T :=$  the set of all triangles

Using only set operations on these three sets, define:

- (a) The set of all equilateral triangles (all sides equal)
- (b) The set of all isosceles triangles (at least two sides equal)
- (c) The set of all scalene triangles (no two sides equal)

**Solution:**

- (a) We require  $a(t) = b(t)$  and  $b(t) = c(t)$  (this obviously implies  $a(t) = c(t)$ ), so the set is  $X \cap Y$
- (b) An isosceles triangle  $t$  can have
  1.  $a(t) = b(t)$ , or
  2.  $b(t) = c(t)$ , or
  3.  $a(t) = c(t)$ .

Now we have assumed that  $a(t)$ ,  $b(t)$ , and  $c(t)$  are in non-decreasing order, so the last condition holds if and only if both the first two do. So the required set is  $X \cup Y$ .

- (c) A scalene triangle has its two smaller sides  $a(t)$  and  $b(t)$  unequal (set  $T \setminus X$ ) and its two larger sides  $b(t)$  and  $c(t)$  unequal (set  $T \setminus Y$ ). Since the sides are listed in the non-decreasing order, either of the above conditions guarantess  $a(t) \neq c(t)$ . So the required set is  $(T \setminus X) \cap (T \setminus Y)$ .

An alternative argument is: A triangle is scalene if and only if it is not isosceles. So using the result of the previous part, the set of scalene triangles is  $T \setminus (X \cup Y)$ . It's easy to confirm that the answers given by the two arguments are actually the same - this is an instance of a general rule called De Morgan's Law.

**Exercise 3** (10 Points) The Fibonacci sequence is defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \text{ for all } n \geq 3 \end{aligned}$$

Prove that

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

**Solution:** The proof proceeds by strong induction. Let  $P(n)$  denote the hypothesis that  $a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$ . For  $n = 1$ ,  $a_1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$ . Thus,  $P(1)$  is true. Similarly,  $P(2)$  is true since  $a_2 = 1$ . Now we assume that the claim holds for all integers between 1 and  $n$  ( $n \geq 3$ ). We need to show that  $P(n+1)$  is true to complete the proof.

From definition,

$$a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 3$$

So,

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\left(\frac{1+\sqrt{5}}{2}\right) + 1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\left(\frac{1-\sqrt{5}}{2}\right) + 1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^2\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}
\end{aligned}$$

This proves the claim for  $n + 1$  and completes the proof by strong induction.

**Exercise 4** (10 Points) Prove by induction:

(a)

$$\sum_{i=1}^n i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2$$

(b)

$$\sum_{i=1}^n i^3 = \left(\sum_{i=1}^n i\right)^2$$

**Solution:**

(a) The proof proceeds by induction. For  $n = 1$ , we have

$$\sum_{i=1}^1 i \cdot 2^i = 2 = 0 \cdot 2^{1+1} + 2,$$

and the claim holds. Now assume that for  $n = k$ ,  $\sum_{i=1}^k i \cdot 2^i = (k-1) \cdot 2^{k+1} + 2$ . Then for  $n = k + 1$ , we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i \cdot 2^i &= \sum_{i=1}^k i \cdot 2^i + (k+1) \cdot 2^{k+1} \\
&= (k-1) \cdot 2^{k+1} + 2 + (k+1) \cdot 2^{k+1} \\
&= (2k) \cdot 2^{k+1} + 2 \\
&= (k) \cdot 2^{k+2} + 2
\end{aligned}$$

Hence the claim holds for  $k + 1$ . This completes the proof by induction.

(b) It is known that  $\sum_{i=1}^n i = n(n+1)/2$ . Thus we need to prove that

$$\sum_{i=1}^n i^3 = (n(n+1)/2)^2$$

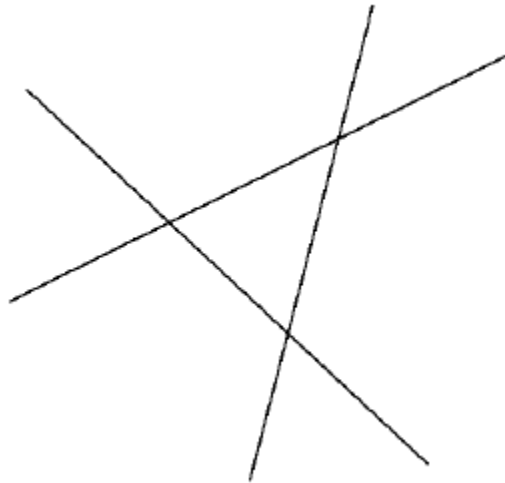
The proof proceeds by induction. For  $n = 1$ , we have  $1^3 = (1(1+1)/2)^2 = 1$ . Assume that for  $n = k$ ,  $\sum_{i=1}^k i^3 = (k(k+1)/2)^2$ . Then for  $n = k+1$ , we have

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= (k(k+1)/2)^2 + (k+1)^3 \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= ((k+1)(k+2)/2)^2 \end{aligned}$$

Hence the claim holds for  $k+1$ . This completes the proof by induction.

**Exercise 5** (10 Points) Consider  $n$  lines in the plane so that no two are parallel and no three intersect in a common point. What is the number of regions into which these lines partition the plane? Prove.

For example, the lines in the following diagram partition the plane into seven regions:



**Solution:** The answer must be some function of  $n$ , which we will denote by  $F(n)$ . We can consider putting down the lines one after the other on the plane (in any arbitrary order), with the total number of regions increasing at each step. After putting the first  $n-1$  lines we have  $F(n-1)$  regions. When we draw the  $n$ th line, we know it must

intersect all of the other  $n - 1$  lines since no two lines are parallel. Since no three share an intersection, this new line has exactly  $n - 1$  intersection points with the other lines. These intersections mean that the new line goes through  $n$  of the existing regions of the plane (at every intersection point, the line leaves one existing region and enters another, and does not change regions before the first intersection, after the last, or between any two intersections). The new line divides each region through which it passes into two, so  $n$  new regions are created. So

$$F(n) = F(n - 1) + n$$

Its easy enough to expand this sum as:

$$\begin{aligned} F(n) &= F(n - 1) + n \\ &= (F(n - 2) + (n - 1)) + n \\ &= (F(n - 3) + (n - 2)) + (n - 1) + n \\ &\cdot \\ &\cdot \\ &\cdot \\ &= F(0) + 1 + 2 + \dots + (n - 1) + n \\ &= F(0) + \frac{n(n + 1)}{2} \text{ (recall the sum of the first } n \text{ natural numbers)} \end{aligned}$$

Now when there are no lines, there is only one region (the whole plane), so  $F(0) = 1$ . So

$$F(n) = \frac{n(n + 1)}{2} + 1.$$

This, however, is not a formal proof! Rather, that was an outline of the kind of reasoning you could use to realize that the right answer is  $n(n + 1)/2 + 1$ . The actual proof proceeds by induction. For the basis, with 0 lines there is 1 region so the claim holds. Now assume it holds for  $n = k$ . When we draw the  $(k + 1)$ -st line, we know it must intersect all of the other  $k$  lines since no lines are parallel. Since no three share an intersection, this new line has exactly  $k$  intersection points with the other lines. These intersections mean that the new line goes through  $k + 1$  of the existing regions of the plane and divides each of those in two, such that  $k + 1$  new regions are created. Adding this to the existing tally gives a total of  $k + 1 + k(k + 1)/2 + 1 = (k^2 + 3k + 2)/2 + 1 = (k + 1)(k + 2)/2 + 1 = (k + 1)((k + 1) + 1)/2 + 1$  lines, so the claim holds for  $k + 1$ . This completes the proof by induction.

**Exercise 6** (10 Points) What's wrong with the following proof?

We prove that for any  $n \in \mathbb{N}$  and any  $a \in \mathbb{R}$ ,  $a^n = 1$ . The proof proceeds by strong induction. For the induction basis,  $a_0 = 1$  and the claim holds. Assume that the claim holds for all  $k$  up to  $n$ . Then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$$

This proves the claim.

**Solution:** In the very first induction step we are assuming that the claim holds for  $n = 0$  and need to prove correctness for  $n = 1$ . At that point the proof assumes that the claim also holds for  $n = -1$ , which was never proved and is not correct in general.

**Exercise 7** (10 Points) Prove the following about strong induction principle, principle of induction, and well ordering:

- (a) Prove the induction principle from the principle of strong induction
- (b) Prove the principle of strong induction from the principle of well ordering
- (c) Prove the principle of well ordering from the induction principle

**Solution:**

- (a) Given a set  $A$  of positive integers, assume the conditions for the induction principle:

$$\cdot 1 \in A \quad \text{--- (1)}$$

$$\cdot \text{If } k \in A, \text{ then } k + 1 \in A \quad \text{--- (2)}$$

hold for set  $A$ . We need to show that  $A = \mathbb{N}^+$  using the strong induction principle. It follows from (1) that  $1 \in A$ . Assume that  $\{1, 2, \dots, n\} \subseteq A$ . This implies that  $n \in A$  and then from the induction premise (2),  $n + 1 \in A$ . Thus by strong induction all positive integers belong to  $A$  and so,  $\mathbb{N}^+ \subseteq A$ . But  $A$  is a set of positive integers and therefore  $A = \mathbb{N}^+$ . This completes the proof.

- (b) Consider the premises of strong induction - a set  $A$  of positive integers exists such that  $1 \in A$  and, if  $\{1, 2, \dots, n\} \subseteq A$ , then  $(n + 1) \in A$  for any  $n \in \mathbb{N}^+$ . Consider the set  $B = \mathbb{N}^+ \setminus A$ , the set of all positive integers not in  $A$ . Assume for the sake of contradiction that  $B$  is nonempty. By the well ordering principle,  $B$  must have a least element. We know that  $1 \notin B$  because one of the induction premises is  $1 \in A$ . Thus there is some least element  $b \in B$  with  $b > 1$ , which implies  $\{b - 1, b - 2, \dots, 1\} \subseteq A$ . But then from the second strong induction premise,  $b = (b - 1) + 1 \in A$ , which contradicts the earlier assertion that  $b \notin A$ . Thus the assumption that  $B$  is nonempty is incorrect, and so  $B = \emptyset$  and  $A = \mathbb{N}^+$ . Thus, assuming the well-ordering principle, if the premises of strong induction hold for a set  $A$  then  $A = \mathbb{N}^+$ , which proves the strong induction principle.
- (c) To simplify things we would first prove that strong induction follows from the induction principle, so proving well-ordering from strong induction will suffice. Given a set  $A$  of positive integers, assume the conditions for the strong induction principle:

- .  $1 \in A$  — (1)
- . If  $\{1, 2, \dots, k\} \subseteq A$ , then  $k + 1 \in A$  — (2)

hold for set  $A$ . We need to show that  $A = \mathbb{N}^+$  using the induction principle. It follows from (1) that  $1 \in A$ . Assume that  $n \in A$ .

Let  $P(n)$  be the following proposition:

$$"\{1, 2, \dots, n\} \subseteq A."$$

For the base case,  $P(1)$  follows trivially from (1). Now assume  $P(n)$  holds, and consider  $P(n + 1)$ . From our induction hypothesis and (2),  $n + 1 \in A$ , or in other words  $P(n)$  implies that  $\{1, 2, \dots, n + 1\} \subseteq A$ , which means  $P(n + 1)$  is true. By the induction principle,  $P(n)$  holds for all positive integers  $n$ , so  $\mathbb{N}^+ \subseteq A$ . But  $A$  is a set of positive integers, so  $A = \mathbb{N}^+$ . This proves the result.

Now we prove the well-ordering principle from strong induction. The proof proceeds by contradiction. Assume there is a nonempty set  $A$  of positive integers without a least element. We will use strong induction to show that this cannot be. Since 1 is the smallest positive integer,  $1 \notin A$  since it would be the least element. Assume that the positive integers from 1 to  $k$  are not in  $A$ . Then if  $(k + 1) \in A$ ,  $(k + 1)$  would be the smallest element so  $(k + 1) \notin A$ . Then by strong induction all positive integers are not in  $A$ , so  $A = \emptyset$ , a contradiction. Therefore the assumption was incorrect and any nonempty set of positive integers must have a least element. Thus the well-ordering principle holds if the induction principle holds.

To conclude, since each principle can be proved from the other, any problem solvable with one can also be solved by the other. Thus the well-ordering principle, induction principle, and the induction principle are equally powerful.

**Exercise 8** (10 Points) Prove the following:

- (a) Prove that  $\sqrt{3}$  is irrational
- (b) Prove that the sum of a rational number and an irrational number results in an irrational number

**Solution:**

- (a) Following the proof that  $\sqrt{2}$  is irrational in the notes, we start by assuming  $\sqrt{3}$  is actually a rational number  $p/q$ , where  $p$  and  $q$  are integers with no common divisor and  $q \neq 0$ , and show this leads to a contradiction. Squaring,

$$3 = p^2/q^2$$

$$\text{or } p^2 = 3q^2$$

Now every integer is of the form  $3n$ ,  $3n + 1$  or  $3n + 2$ , and the squares in the three cases are  $9n^2$ ,  $9n^2 + 6n + 1$  and  $9n^2 + 12n + 4$ . The first is divisible by 3 and the second and third leave a remainder of 1 when divided by 3. So a perfect square is a multiple of 3 if and only if its square root is divisible by 3. In other words,  $p$  must be divisible by 3, say  $p = 3k$  for some integer  $k$ . But then

$$9k^2 = 3q^2$$

$$\text{or, } q^2 = 3p^2$$

and by the same argument,  $q$  must also be divisible by 3. This contradicts our assumption that  $p$  and  $q$  have no common divisor, and so  $\sqrt{3}$  must, in fact, be irrational.

- (b) The proof proceeds by contradiction. Assume that the sum of a rational number and an irrational number results in a rational number. Let us denote the irrational number by  $x$  and the other two rational numbers as  $a/b$  and  $c/d$  respectively, where  $a, b, c, d$  are integers and  $b, d \neq 0$ . Now, from our assumption

$$a/b + x = c/d$$

$$\text{or, } x = c/d - a/b$$

$$\text{or, } x = \frac{cb - ad}{bd}$$

which is rational. But this contradicts our assumption that  $x$  is irrational. This completes the proof.

**Exercise 9** (10 Points) Another general proof technique is *proof by contrapositive*. To prove a statement of the form “If A, then B” it suffices to prove “If not B, then not A”. Using the contrapositive technique, prove that if the product of integers  $p$  and  $q$  is odd, then both  $p$  and  $q$  must be odd.

**Solution:** We prove this by the contrapositive method. Thus we need to prove that if not both  $p$  and  $q$  are odd, then their product  $pq$  is not odd (i.e. it must be even). Let  $p = 2k$  (even) and  $q = 2l + 1$  (odd), for some  $k, l \in \mathbb{Z}$ . Now the product  $pq = 2k \cdot (2l + 1) = 4kl + 2k = 2(2kl + k)$ , is even. Similarly, for the case when both  $p$  and  $q$  are even,  $pq$  is again even. Hence, if not both  $p$  and  $q$  are odd, then their product  $pq$  is not odd. This completes the proof by contrapositive method.

**Exercise 10** (10 Points) Given that  $a$ ,  $b$ , and  $c$  are odd integers, prove that  $ax^2 + bx + c = 0$  does not have a rational root.

**Solution:** The proof proceeds by contradiction. Assume that  $ax^2 + bx + c = 0$  does have a rational root  $m/n$ , where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ , when  $a$ ,  $b$ , and  $c$  are odd integers.



We can assume without the loss of generality that  $m$  and  $n$  do not have any common factors. The equation can be re-written as -

$$a(m/n)^2 + b(m/n) + c = 0$$

$$\text{or, } am^2 + bmn + cn^2 = 0$$

Let us first consider the case when  $m$  is even. Then  $n$  is odd, since otherwise  $m$  and  $n$  will have 2 as a common factor. Now because square of an even number is even and square of an odd number is odd,  $am^2 + bmn$  is even and  $cn^2$  is odd. Hence  $am^2 + bmn + cn^2$  cannot be 0.

Next let us consider the case when  $m$  is odd. If  $n$  is even, a similar argument as above can be used to show that  $am^2 + bmn + cn^2$  cannot be 0. If  $n$  is odd, then all of  $am^2$ ,  $bmn$  and  $cn^2$  are odd. However, the sum of three odd numbers cannot be 0. Thus, we have arrived at a contradiction and  $m/n$  cannot be a root of the equation. This completes the proof.