CS103X: Discrete Structures
Homework Assignment 4

Due February 22, 2008

Exercise 1 (10 points)**.** Silicon Valley questions:

(a) How many possible six-figure salaries (in whole dollar amounts) are there that contain at least three distinct digits?

(b) Second Silicon Valley question: What is the number of six-figure salaries that are not multiples of either 3, 5, or 7.

Exercise 2 (15 points)**.** A rook on a chessboard is said to put another chess piece under attack if they are in the same row or column.

(a) How many ways are there to arrange 8 rooks on a chessboard (the usual 8×8 one) so that none are under attack?

(b) How many ways are there to arrange k rooks on an $n \times n$ chessboard so that none are under attack?

(c) Imagine a three-dimensional chess variant played on a $8 \times 8 \times 8$ board. (512 cells overall.) Call it Weir-D Chess. A *battleship* is a Weir-D Chess piece that can attack any piece that is in the same two-dimensional layer, along some coordinate. (For example, a battleship in position $(5, 2, 6)$ puts cell $(8, 2, 1)$ under attack, but not cell $(8, 3, 1)$.) How many ways are there to arrange 8 battleships on a Weir-D Chess board so that none are under attack?

Give solutions with no summation.

Exercise 3 (15 points)**.** A function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ is called monotone nondecreasing if $1 \leq i < j \leq n \;\Rightarrow\; f(i) \leq f(j)$.

(a) How many such functions are there?

(b) How many such functions are there that are surjective?

(c) How many such functions are there that are injective?

Exercise 4 (10 points)**.** How many ways are there to express a positive integer n as:

(a) A sum of k natural numbers? (For example, if $n = 2$ and $k = 3$ the answer is 6, since $2 = 2+0+0 = 0+2+0 = 0+0+2 = 1+1+0 = 1+0+1 = 0+1+1$.)

(b) A sum of positive integers?

The order of the summands is important. (Imagine the summation written down.)

Exercise 5 (10 points)**.** Prove either algebraically or combinatorially:

(a) For $p, n \geq 0$, $\displaystyle\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}$

(b) $\displaystyle\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$

Exercise 6 (10 points)**.** Give a closed-form expression (without summation) for the following:

$$\sum_{k=0}^{n} 2^k \binom{n}{k}.$$

Exercise 7 (10 points)**.** In a mathematics contest with three problems, 80% of the participants solved the first problem, 75% solved the second and 70% solved the third. Prove that at least 25% of the participants solved all three problems. (The claim might seem obvious — find a *proof.*)

Exercise 8 (10 points)**.** What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 = 50,$$

such that $0 \leq x_i \leq 20$ for each $1 \leq i \leq 3$?

Exercise 9 (10 points)**.** There are n people at a party, and each person has arrived in a different hat. The revelry leaves them slightly tipsy, so each of them goes home wearing someone else's hat. Find the number of ways of putting n hats on n people so that no person is wearing his/her own hat. *Give the full proof.*