CS103X: Discrete Structures
Homework Assignment 5

Due February 29, 2008

Exercise 1 (20 points)**.** Prove:

(a) In a collection of $n+1$ integers, at least two are congruent modulo n.

(b) Among any five points with integer coordinates in the plane, there are two, such that the center of the line segment that connects them also has integer coordinates.

(c) At a cocktail party with at least two people, there are two that know the same number of party attendees.

(d) Among five points chosen from within a square with side length 1, at least two lie within distance $\sqrt{2}/2$.

Exercise 2 (10 points)**.** A stressed-out computer science professor consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days exactly 100 espressos were consumed.

Exercise 3 (10 points)**.** A betting game is played by filling out a number of betting cards. Each time you fill out a card, you pick 4 natural numbers ranging from 1 to 16. After filling out some number of cards, you hand them in and the house picks 4 of the numbers to be "losers" at random. A card is a winner if it has none of those "loser" numbers. Prove that if you fill out 6 cards, it is always possible that none of them will win.

Exercise 4 (10 points)**.** Prove that if every point on a line is painted cardinal or white, there exists three points of the same color such that one is the midpoint of the line segment formed by the other two.

Exercise 5 (10 points)**.** Prove that at a cocktail party with ten or more people, there are either three mutual acquaintances or four mutual strangers.

Exercise 6 (20 points)**.** Order the following functions by asymptotic growth rates. That is, list them as $f_1(n), f_2(n), \ldots, f_{10}(n)$, such that $f_i(n) = O(f_{i+1}(n))$ for all i. Give a brief argument to support the inequality $f_i(n) = O(f_{i+1}(n))$ for every i. (You don't have to give all the details.)

$$n^2, \log_{10} n, n \log_2 n, n^{\ln n}, (\ln n)^n, \ln(n^n), (\ln n)^{\ln n}, (\ln n)^{\ln \ln n}, (\ln \ln n)^{\ln n}, 3^{n/2}.$$

Hint: Take logarithms to bring the exponents down. Use basic properties of logarithms.

Exercise 7 (10 points)**.** What's wrong with the following induction proof?

> Define $f(n) = n^2$. We prove by induction that $f(n) = O(n)$. When $n = 1$, $1^2 = 1$ and the claim holds. Assume that $k^2 = O(k)$ for some $k \geq 1$. Then $(k+1)^2 = k^2 + 2k + 1$. Each of the summands on the right side is $O(k)$, and thus $(k+1)^2 = O(k)$. This completes the proof by induction.

Exercise 8 (10 points)**.** Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$. Is $\propto$ an equivalence relation? A partial order? A total order? Prove.