CS103X: Discrete Structures
Homework Assignment 7

Due March 14, 2008

Exercise 1 (20 points)**.** Let G be a graph that has no induced subgraphs that are P_4 or C_3.

(a) Prove that G is bipartite.

Solution Since we know a graph is bipartite if and only if it has no odd cycles, we can equivalently prove that G has no odd cycles. Then by taking the contrapositive, it is equivalent to prove that any graph with an odd cycle has either P_4 or C_3 as an induced subgraph. From here we proceed by contradiction, assume there exists some graph G with an odd cycle and no induced subgraphs that are P_4 or C_3. Obviously, if this cycle is length 3, then C_3 is an induced subgraph. If the length is 5 or greater, select any four adjacent points in the cycle (i.e. points A, B, C, D such that edges AB, BC, CD are part of the cycle) and consider the induced subgraph on those four. If the cycle edges are the only ones present, P_4 is an induced subgraph. If not, then one of the edges AC, BD, AD must be in the original graph. If AC is, then we have an induced C_3 subgraph on A, B, C, and if BD is then there is an induced C_3 on B, C, D. The only remaining possibility is that AD is present; then we can make a new cycle by removing AB, BC, CD from the original cycle and adding AD. This new cycle has length 2 less than the original, so it is still odd. Since this is the only possibility that does not immediately produce a C_3 or P_4, we can repeat this process to examine progressively smaller cycles. But then eventually we will create a cycle of length 3, the shortest possible odd-cycle length, which produces a C_3 induced subgraph. This is a contradiction, and thus any graph with an odd cycle must have either P_4 or C_3 as an induced subgraph. This completes the proof.

(**Note:** The word "induced" is very important here! See the definition of "induced subgraph" in the lecture notes — it is *not* the same as a "subgraph".)

(b) Assume in addition that G is connected. Prove that G is a complete bipartite graph.

Solution Suppose vertex u in one of the classes is not connect to vertex v in the other class. Since G is connected, there exists some shortest path from u to v. Consider the induced subgraph produced by this path. This path must be of at least length 3 since they live in different classes and are not connected. If it is of length 3, then we have a subgraph of P_4. Otherwise if it has length > 3, consider the first four vertices of the path $\{u, u_1, u_2, u_3\}$. This must be the shortest path from u to u_3 since it is in the shortest path between u to v (otherwise there would be a shorter path from u to v). Thus, if we consider this induced subgraph, we again get a subgraph of P_4. Thus, every u, v pair must be connected, which implies that the bipartite graph is complete.

Exercise 2 (15 points)**.** Given a bipartite graph G, prove that its two classes are unique (up to interchanging their order) if and only if G is connected.

Solution If u and v are in different partite sets in some bipartition, and they are connected by a path, then the path must have odd length. Also, if an odd length path connects two vertices, then any bipartition must put them in different partite sets (since every edge switches sets). Putting these together, we conclude that every pair originally separated must remain separated in any other bipartition, giving the result. (Only if) Each connected component of a bipartite graph is bipartite. A disconnected graph has more than one connected component. Swap the partite sets in exactly one connected component to get a different bipartition.

Exercise 3 (15 points)**.** For any $k \in \mathbb{N}^+$, prove that a k-regular bipartite graph has a perfect matching.

Solution We will apply **Theorem 15.3.4** from the lecture notes. So we need to show that for the two classes, A and B, that $|A| = |B|$ and $|\Gamma(S)| \geq |S| \ \forall S \subseteq A$. First note that there must be the same number of vertices on each class otherwise there are more edges leaving one class than there are entering the other class. Now consider any set $S \subseteq A$. Each vertex has degree k, so if $|S| < k$ then $|\Gamma(S)| \geq |S|$ trivially holds. In the other case, $|S| \geq k$, we will show that $|\Gamma(S)| \geq |S|$ by contradiction. Suppose $|\Gamma(S)| < |S|$. There are $k|S|$ edges leaving set S and they all enter the set $\Gamma(S)$. However, there can only be $k|\Gamma(S)| < k|S|$ edges connected to the set, meaning not all of the edges leaving S is enclosed in $\Gamma(S)$. This is a contradiction. Therefore, $|\Gamma(S)| \geq |S| \ \forall S \subseteq A$ also holds. By **Theorem 15.3.4** from the lecture notes, we know that there must be a perfect matching.

Exercise 4 (15 points)**.** Given a tree G that contains a vertex of degree k, prove that G has at least k leaves.

Solution Let vertex v be in a tree $G = (V, E)$ with degree k. Consider the induced subgraph G' by taking out vertex v which is k connected components, each being a tree. Formally, $G' = (V \setminus v, E')$ where $E' = E \setminus \{\{v, u\} : \{v, u\} \in E \ \forall u \in V\}$. First note that if any connected component only has 1 vertex, then it was a leaf in G. Consider the connected components that have more than one vertex. By **Lemma 16.1.3** the connected component has at least 2 leaves. Since in G, it is connected to v so we have lost 1 leaf, so each connected component contributes at least 1 leaf. Therefore, for each connected component, it contributes either 1 leaf (for 1 vertex component) or at least 1 leaf (for $>$ 1 vertex component). Therefore, with k components, we have at least k leaves.

Exercise 5 (15 points)**.** Prove that $G = (V, E)$ is a tree if and only if $|V| = |E| + 1$ and G has no cycles.

Solution We will use induction on $|V|$. For a tree with a single vertex, the claim holds since $|E| + 1 = 0 + 1 = 1$. Now suppose that the claim holds for all n-vertex trees and consider an (n + 1)-vertex tree. Let v be a leaf of the tree. Deleting v and its incident edge gives a smaller tree for which the equation $|V| = |E| + 1$ holds by induction. If we add back the vertex v and its incident edge, then the equation still holds because the number of vertices and number of edges both increased by 1. Thus, the claim holds for the $n+1$-vertex tree and, by induction, for all trees.

Exercise 6 (20 points)**.** Let G be a simple graph with n vertices and k connected components.

(a) What is the minimum possible number of edges of G?

Solution Let each component i have c_i vertices. If we put a minimum spanning tree to keep it connected, we get $c_i - 1$ edges. So the total number of edges is

$$\sum_{i=1}^{k}(c_i - 1) = \sum_{i=1}^{k} c_i - k = n - k$$

Thus it does not matter how the components are selected, we always get this minimum.

(b) What is the maximum possible number of edges of G?

Solution Let each component i have c_i vertices. If we put a complete graph for each connected component, we will maximize edges. So the total number of edges is

$$\sum_{i=1}^{k}\binom{c_i}{2} = \sum_{i=1}^{k}\frac{c_i(c_i-1)}{2} = \frac{1}{2}(\sum_{i=1}^{k} c_i^2 - c_i) = \frac{1}{2}(\sum_{i=1}^{k} c_i^2 - n)$$

We now need to find some distribution of vertices for each connected component such that we maximize this expression. Consider some sequence $\{c_1, c_2, \ldots, c_k\}$ such that $c_1 \leq c_2 \leq \ldots \leq c_k$. Let's compare the number of edges produced with sequence $\{c_1 - 1, c_2 + 1, c_3, \ldots, c_k\}$. Notice that this sequence is still in increasing order. The additional edges gained from using this new sequence for number of vertices for each of the components:

$$(c_1 - 1)^2 + (c_2 + 1)^2 - c_1^2 - c_2^2 = 2(c_2 - c_1 + 1)$$

So this is a positive increase as long as $c_1 \leq c_2 + 1$. We have assumed that $c_1 \leq c_2$, so this means that decreasing c_1 by 1 and increasing c_2 by 1 results in creation of additional edges. We can apply this argument to any two consecutive c_i and c_j repeatedly, thus resulting in $c_1 = c_2 = \ldots = c_{k-1} = 1$ and $c_k = n - (k-1)$. Therefore the maximal number of edges that can be created is

$$\begin{aligned} &\frac{1}{2}\Big(k - 1 + (n - (k-1))^2 - n\Big) \\ &= \frac{1}{2}\Big((n-k)^2 + (n-k)\Big) \end{aligned}$$