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Date: 7/12/21  
Lecture Topics: 4.1 Continuous Random Variable Basics

[Tags: PDFs, CDFs]

1. Alex decided he wanted to create a “new” type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We’ll denote a random variable $X$ having the “Uniform-2” distribution as $X \sim \text{Unif}(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

Here’s an example of $\text{Unif}(2, 4, 8, 12)$:

a. Find the PDF $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. Use a piecewise function.

b. Find the CDF $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. Use a piecewise function.

**Solution:**

a. We need to just find the height $h$ which is constant. The area under the curve is two rectangles, and we need it to be 1, so

$$(b - a)h + (d - c)h = 1 \rightarrow h = \frac{1}{(b - a) + (d - c)}$$

Hence,

$$f_X(x) = \begin{cases} 
\frac{1}{(b - a) + (d - c)}, & x \in [a, b] \cup [c, d] \\
0, & \text{otherwise}
\end{cases}$$

b. The CDF is the cumulative area to the left of a certain point. We have 5 cases.

If $x < a$, the probability $X \leq x$ is 0, since there’s no area to the left or no probability of this happening.

If $x \geq d$, the probability $X \leq x$ is 1, since it is guaranteed that $X \leq d$ (the total area to the left is 1).

If $b \leq x < c$, the probability $X \leq x$ is just the area of the left rectangle, which is base * height or
If \( a \leq x < b \), then we have a smaller subrectangle of base \( x - a \), multiplied by height, which is

\[
\frac{(x - a) \cdot 1}{(b - a) + (d - c)}.
\]

If \( c \leq x < d \), we have the entire left rectangle, and a smaller subrectangle of base \( x - c \), which is

\[
\frac{(x - c) \cdot 1}{(b - a) + (d - c)} + (b - a) \cdot \frac{1}{(b - a) + (d - c)}
\]

So, we have:

\[
F_X(x) = \begin{cases} 
  0, & x < a \\
  \frac{x - a}{(b - a) + (d - c)}, & a \leq x < b \\
  \frac{b - a}{(b - a) + (d - c)}, & b \leq x < c \\
  \frac{(x - c) + (b - a)}{(b - a) + (d - c)}, & c \leq x < d \\
  1, & x \geq d
\end{cases}
\]

**Solution:** Watch lecture 😊