1 Useful identities related to summations

Since it may have been a while since some folks have worked with summations, I just wanted to
provide a reference on them that you may find useful in your future work. Here are some useful
identities and rules related to working with summations. In the rules below, \( f \) and \( g \) are arbitrary
real-valued functions.

Pulling a constant out of a summation:
\[
\sum_{n=s}^t C \cdot f(n) = C \cdot \sum_{n=s}^t f(n), \text{ where } C \text{ is a constant.}
\]

Eliminating the summation by summing over the elements:
\[
\sum_{i=1}^n x = nx
\]
\[
\sum_{i=m}^n x = (n - m + 1)x
\]
\[
\sum_{i=s}^n f(C) = (n - s + 1)f(C), \text{ where } C \text{ is a constant.}
\]

Combining related summations:
\[
\sum_{n=s}^j f(n) + \sum_{n=j+1}^t f(n) = \sum_{n=s}^t f(n)
\]
\[
\sum_{n=s}^t f(n) + \sum_{n=s}^t g(n) = \sum_{n=s}^t [f(n) + g(n)]
\]

Changing the bounds on the summation:
\[
\sum_{n=s}^t f(n) = \sum_{n=s+p}^{t+p} f(n - p)
\]
"Reversing" the order of the summation:

\[ \sum_{n=a}^{b} f(n) = \sum_{n=b}^{a} f(n) \]

Arithmetic series:

\[ \sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]  
(with a moment of silence for C. F. Gauss.)

\[ \sum_{i=m}^{n} i = \frac{(n-m+1)(n+m)}{2} \]

Arithmetic series involving higher order polynomials:

\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \]

\[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[ \sum_{i=1}^{n} i \right]^2 \]

Geometric series:

\[ \sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x} \]

\[ \sum_{i=m}^{n} x^i = \frac{x^{n+1}-x^m}{x-1} \]

\[ \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1 \]

More exotic geometric series:

\[ \sum_{i=0}^{n} i2^i = 2 + 2^{n+1}(n - 1) \]

\[ \sum_{i=0}^{n} \frac{i}{2^i} = \frac{2^{n+1} - n - 2}{2^n} \]
Taylor expansion of exponential function:

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]

Binomial coefficient:

\[ \sum_{i=0}^{n} \binom{n}{i} = 2^n \]

Much more information on binomial coefficients is available in the Ross textbook.

2 Growth rates of summations

Besides solving a summation explicitly, it is also worthwhile to know some general growth rates on sums, so you can (tightly) bound a sum if you are trying to prove something in the big-Oh/Theta world. If you’re not familiar with big-Theta (\( \Theta \)) notation, you can think of it like big-Oh notation, but it actually provides a “tight” bound. Namely, big-Theta means that the function grows no more quickly and no more slowly than the function specified, up to constant factors, so it’s actually more informative than big-Oh.

Here are some useful bounds:

\[ \sum_{i=1}^{n} i^c = \Theta(n^{c+1}), \text{ for } c \geq 0. \]
\[ \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n) \]
\[ \sum_{i=1}^{n} e^i = \Theta(e^n), \text{ for } c \geq 2. \]

3 A few identities related to products

Recall that the mathematical symbol \( \prod \) represents a product of terms (analogous to \( \Sigma \) representing a sum of terms). Below, we give some useful identities related to products.

Definition of factorial:

\[ \prod_{i=1}^{n} i = n! \]

Note that 0! = 1 by definition.

Stirling’s approximation for \( n! \) is given below. This approximation is useful when computing \( n! \) for large values of \( n \) (particularly when \( n > 30 \)).

\[ n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \text{ or equivalently } n! \approx \sqrt{2\pi n}^{(n+\frac{1}{2})}e^{-n} \]
Eliminating the product by multiplying over the elements:

\[ \prod_{i=1}^{n} C = C^n, \text{ where } C \text{ is a constant.} \]

Combining products:

\[ \prod_{i=1}^{n} f(i) \prod_{i=1}^{n} g(i) = \prod_{i=1}^{n} f(i) \cdot g(i) \]

Turning products into summations (by taking logarithms, assuming \( f(i) > 0 \) for all \( i \)):

\[ \log \left( \prod_{i=1}^{n} f(i) \right) = \sum_{i=1}^{n} \log f(i) \]

4 \hspace{1em} \textbf{Suggestions for computing permutations and combinations}

For your problem set solutions it is fine for your answers to include factorials, exponentials, or combinations; you don’t need to calculate those all out to get a single numeric answer. However, if you’d like to work with those in Python, R, or Microsoft Excel, here are a few functions you may find useful.

In Python:

\begin{verbatim}
math.factorial(n) computes n!
scipy.special.binom(n, m) computes \( \binom{n}{m} \) (as a float)
math.exp(n) computes \( e^n \)
n ** m computes \( n^m \)
\end{verbatim}

Names to the left of the dots (.) are modules that need to be imported before being used: import math, scipy.special.
In R:

\[ \text{factorial(n)} \] computes \( n! \)
\[ \text{choose(n, m)} \] computes \( \binom{n}{m} \)
\[ \text{exp(n)} \] computes \( e^n \)
\[ n^m \] computes \( n^m \)

In Microsoft Excel:

\[ \text{FACT(n)} \] computes \( n! \)
\[ \text{COMBIN(n, m)} \] computes \( \binom{n}{m} \)
\[ \text{EXP(n)} \] computes \( e^n \)
\[ \text{POWER(n, m)} \] computes \( n^m \)

To use functions in Excel, you need to set a cell to equal a function value. For example, to compute \( 3! \cdot \binom{5}{2} \), you would put the following in a cell:

\[ = \text{FACT(3)} * \text{COMBIN(5, 2)} \]

Note the equals sign (=) at the beginning of the expression.
5 A little review of calculus
Since it may have been a while since you did calculus, here are a few rules that you might find useful.

Product Rule for derivatives:
\[ d(u \cdot v) = du \cdot v + u \cdot dv \]

Derivative of exponential function:
\[ \frac{d}{dx} e^u = e^u \frac{du}{dx} \]

Integral of exponential function:
\[ \int e^u \, du = e^u \]

Derivative of natural logarithm:
\[ \frac{d}{dx} \ln(x) = \frac{1}{x} \]

Integral of 1/x:
\[ \int \frac{1}{x} \, dx = \ln(x) \]

Integration by parts (everyone’s favorite!):

Choose a suitable u and dv to decompose the integral of interest:
\[ \int u \cdot dv = u \cdot v - \int v \cdot du \]

Here’s the underlying rule that integration by parts is derived from:
\[ \int d(u \cdot v) = u \cdot v = \int du \cdot v + \int u \cdot dv \]

6 Bibliography
Additional information on sums and products can generally be found in a good calculus or discrete mathematics book. The discussion of summations above is based on Wikipedia (http://en.wikipedia.org/wiki/Summation).