

## Combinatorics

Handout by Chris Piech, with examples by Mehran Sahami

As we mentioned last class, the principles of counting are core to probability. Counting is like the foundation of a house (where the house is all the great things we will do later in CS109, such as machine learning). Houses are awesome. Foundations, on the other hand, are pretty much just concrete in a hole. But don't make a house without a foundation. It won't turn out well.

### Permutations

**Permutation Rule:** A permutation is an ordered arrangement of  $n$  distinct objects. Those  $n$  objects can be permuted in  $n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1 = n!$  ways.

This changes slightly if you are permuting a subset of distinct objects, or if some of your objects are indistinct. We will handle those cases shortly!

#### *Example 1*

**Part A:** iPhones have 4-digit passcodes. Suppose there are 4 smudges over 4 digits on the screen. How many distinct passcodes are possible?

**Solution:** Since the order of digits in the code is important, we should use permutations. And since there are exactly four smudges we know that each number is distinct. Thus, we can plug in the permutation formula:  $4! = 24$ .

**Part B:** What if there are 3 smudges over 3 digits on screen?

**Solution:** One of 3 digits is repeated, but we don't know which one. We can solve this by making three cases, one for each digit that could be repeated (each with the same number of permutations). Let  $A, B, C$  represent the 3 digits, with  $C$  repeated twice. We can initially pretend the two  $C$ 's are distinct. Then each case will have  $4!$  permutations:

$$A \ B \ C_1 \ C_2$$

However, then we need to eliminate the double-counting of the permutations of the identical digits (one  $A$ , one  $B$ , and two  $C$ 's):

$$\frac{4!}{2! \cdot 1! \cdot 1!}$$

Adding up the three cases for the different repeated digits gives

$$3 \cdot \frac{4!}{2! \cdot 1! \cdot 1!} = 3 \cdot 12 = 36$$

**Part C:** What if there are 2 smudges over 2 digits on the screen?

**Solution:** There are two possibilities: 2 digits used twice each, or 1 digit used 3 times, and other digit used once.

$$\frac{4!}{2! \cdot 2!} + 2 \cdot \frac{4!}{3! \cdot 1!} = 6 + (2 \cdot 4) = 6 + 8 = 14$$

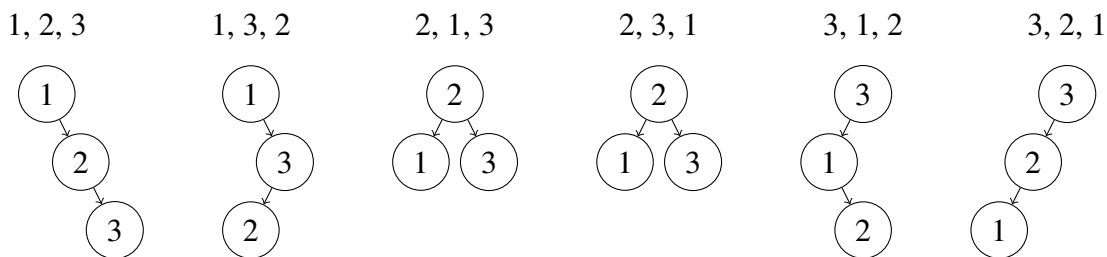
**Example 2**

Recall the definition of a **binary search tree (BST)**, which is a binary tree that satisfies the following three properties for *every* node  $n$  in the tree:

1.  $n$ 's value is greater than all the values in its left subtree.
2.  $n$ 's value is less than all the values in its right subtree.
3. both  $n$ 's left and right subtrees are binary search trees.

**Problem:** How many possible binary search trees are there which contain the three values 1, 2, and 3, and have a degenerate structure (i.e., each node in the BST has at most one child)?

**Solution:** We start by considering the fact that the three values in the BST (1, 2, and 3) may have been inserted in any one of  $3!$  (=6) orderings (permutations). For each of the  $3!$  ways the values could have been ordered when being inserted into the BST, we can determine what the resulting structure would be and determine which of them are degenerate. Below we consider each possible ordering of the three values and the resulting BST structure.



We see that there 4 degenerate BSTs here (the first two and last two).

**Permutations of Indistinct Objects**

**Permutation of Indistinct Objects:** Generally when there are  $n$  objects and

$n_1$  are the same (indistinguishable),

$n_2$  are the same,

...

and  $n_r$  are the same,

then there are

$$\frac{n!}{n_1!n_2! \dots n_r!}$$

distinct permutations of the objects.

### Example 3

**Problem:** How many distinct bit strings can be formed from three 0's and two 1's?

**Solution:** 5 total digits would give 5! permutations. But that is assuming the 0's and 1's are distinguishable (to make that explicit, let's give each one a subscript). Here is a subset of the permutations.

0 <sub>1</sub>	1 <sub>1</sub>	1 <sub>2</sub>	0 <sub>2</sub>	0 <sub>3</sub>
0 <sub>1</sub>	1 <sub>1</sub>	1 <sub>2</sub>	0 <sub>3</sub>	0 <sub>2</sub>
0 <sub>2</sub>	1 <sub>1</sub>	1 <sub>2</sub>	0 <sub>1</sub>	0 <sub>3</sub>
0 <sub>2</sub>	1 <sub>1</sub>	1 <sub>2</sub>	0 <sub>3</sub>	0 <sub>1</sub>
0 <sub>3</sub>	1 <sub>1</sub>	1 <sub>2</sub>	0 <sub>1</sub>	0 <sub>2</sub>
0 <sub>3</sub>	1 <sub>1</sub>	1 <sub>2</sub>	0 <sub>2</sub>	0 <sub>1</sub>

If identical digits are indistinguishable, then all the listed permutations are the same. For any given permutation, there are 3! ways of rearranging the 0's and 2! ways of rearranging the 1's (resulting in indistinguishable strings). We have over-counted. Using the formula for permutations of indistinct objects, we can correct for the over-counting:

$$\text{Total} = \frac{5!}{3! \cdot 2!} = \frac{160}{6 \cdot 2} = \frac{120}{12} = 10.$$

### Combinations

**Combinations:** A combination is an unordered selection of  $r$  objects from a set of  $n$  objects. If all objects are distinct, then the number of ways of making the selection is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

This is often read as “ $n$  choose  $r$ ”.

Consider this general way to produce combinations: To select  $r$  unordered objects from a set of  $n$  objects, e.g., “7 choose 3”,

1. First consider permutations of all  $n$  objects. There are  $n!$  ways to do that.
2. Then select the first  $r$  in the permutation. There is one way to do that.
3. Note that the order of  $r$  selected objects is irrelevant. There are  $r!$  ways to permute them. The selection remains unchanged.
4. Note that the order of  $(n - r)$  unselected objects is irrelevant. There are  $(n - r)!$  ways to permute them. The selection remains unchanged.

$$\text{Total} = \frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r} \quad \text{e.g., } \frac{7!}{3!4!} = 35$$

which is the combinations formula.

**Example 4**

**Problem:** In the Hunger Games, how many ways are there of choosing 2 villagers from district 12, which has a population of 8,000?

**Solution:** This is a straightforward combinations problem.  $\binom{8000}{2} = 31,996,000$ .

**Example 5**

**Part A:** How many ways are there to select 3 books from a set of 6?

**Solution:** If each of the books are distinct, then this is another straightforward combination problem. There are  $\binom{6}{3} = \frac{6!}{3!3!} = 20$  ways.

**Part B:** How many ways are there to select 3 books if there are two books that should not both be chosen together (for example, don't choose both the 8th and 9th edition of the Ross textbook).

**Solution:** This problem is easier to solve if we split it up into cases. Consider the following three different cases:

Case 1: Select the 8th Ed. and 2 other non-9th Ed.: There are  $\binom{4}{2}$  ways of doing so.

Case 2: Select the 9th Ed. and 2 other non-8th Ed.: There are  $\binom{4}{2}$  ways of doing so.

Case 3: Select 3 from the books that are neither the 8th nor the 8th edition: There are  $\binom{4}{3}$  ways of doing so.

Using our old friend the Sum Rule of Counting, we can add the cases:

$$\text{Total} = 2 \cdot \binom{4}{2} + \binom{4}{3} = 16.$$

Alternatively, we could have calculated all the ways of selecting 3 books from 4, and then subtract the “forbidden” ones (i.e., the selections that break the constraint). Chris Piech calls this the Forbidden City method.

Forbidden Case: Select 8th edition and 9th edition and 1 other book. There are  $\binom{4}{1}$  ways of doing so (which equals 4).

$$\text{Total} = \text{All possibilities} - \text{forbidden} = 20 - 4 = 16.$$

Two different ways to get the same right answer!

**Bucketing / Group Assignment**

You have probably heard about the dreaded “balls and urns” probability examples. What are those all about? They are for counting the many different ways that we can think of stuffing elements into containers. (It turns out that Jacob Bernoulli was into voting and ancient Rome. And in ancient Rome they used urns for ballot boxes.) This “bucketing” or “group assignment” process is a useful metaphor for many counting problems.

**Example 6**

**Problem:** Say you want to put  $n$  distinguishable balls into  $r$  urns. (No! Wait! Don't say that!) Okay, fine. No urns. Say we are going to put  $n$  strings into  $r$  buckets of a hash table where all outcomes are equally likely. How many possible ways are there of doing this?

**Solution:** You can think of this as  $n$  independent experiments each with  $r$  outcomes. Using our friend the General Principle of Counting, this comes out to  $r^n$ .

**Divider Method:** A divider problem is one where you want to place  $n$  indistinguishable items into  $r$  containers. The divider method works by imagining that you are going to solve this problem by sorting two types of objects, your  $n$  original elements and  $(r - 1)$  dividers. Thus, you are permuting  $n + r - 1$  objects,  $n$  of which are same (your elements) and  $r - 1$  of which are same (the dividers). Thus:

$$\text{Total ways} = \frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{n} = \binom{n+r-1}{r-1}.$$

**Example 7**

**Part A:** Say you are a startup incubator and you have \$10 million to invest in 4 companies (in \$1 million increments). How many ways can you allocate this money?

**Solution:** This is just like putting 10 balls into 4 urns. Using the Divider Method we get:

$$\text{Total ways} = \binom{10+4-1}{10} = \binom{13}{10} = 286.$$

**Part B:** What if you don't have to invest all \$10 M? (The economy is tight, say, and you might want to save your money.)

**Solution:** Imagine that you have an extra company: yourself. Now you are investing \$10 million in 5 companies. Thus, the answer is the same as putting 10 balls into 5 urns.

$$\text{Total ways} = \binom{10+5-1}{10} = \binom{14}{10} = 1001.$$

**Part C:** What if you know you want to invest at least \$3 million in Company 1?

**Solution:** There is one way to give \$3 million to Company 1. The number of ways of investing the remaining money is the same as putting 7 balls into 4 urns.

$$\text{Total ways} = \binom{7+4-1}{7} = \binom{10}{7} = 120.$$