

Probability

It is that time in the quarter (it is still week one) when we get to talk about probability. Again we are going to build up from first principles. We will heavily use the counting that we learned earlier this week.

Event Space and Sample Space

Sample space, S , is set of all possible outcomes of an experiment. For example:

1. Coin flip: $S = \{\text{Head}, \text{Tails}\}$
2. Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
3. Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
4. # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ (non-neg. ints)
5. YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$

Event Space, E , is some subset of S that we ascribe meaning to. In set notation ($E \subseteq S$).

1. Coin flip is heads: $E = \{\text{Head}\}$
2. ≥ 1 head on 2 coin flips: $E = \{(H, H), (H, T), (T, H)\}$
3. Roll of die is 3 or less: $E = \{1, 2, 3\}$
4. # emails in a day ≤ 20 : $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
5. Wasted day (≥ 5 YT hrs.): $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

Probability

In the 20th century humans figured out a way to precisely define what a probability is:

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

In English this reads: lets say you perform n trials of an experiment. The probability of a desired event E is the ratio of trials that result in E to the number of trials performed (in the limit as your number of trials approaches infinity).

That is mathematically rigorous. You can also apply other semantics to the concept of a probability. One common meaning ascribed is that $P(E)$ is a measure of the chance of E occurring.

I often think of a probability in another way: I don't know everything about the world. So it goes. As a result I have to come up with a way of expressing my belief that E will happen given my limited knowledge. This interpretation acknowledges that there are two sources of probabilities: natural randomness and our own uncertainty.

Axioms of Probability

Here are some basic truths about probabilities:

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F mutually exclusive ($E \cap F = \emptyset$),
then $P(E) + P(F) = P(E \cup F)$

You can convince yourself of the first axiom by thinking about the math definition of probability. As you perform trials of an experiment it is not possible to get more events than trials (thus probabilities are less than 1) and its not possible to get less than 0 occurrences of the event.

The second axiom makes sense too. If your event space *is* the sample space, then each trial must produce the event. This is sort of like saying; the probability of you eating cake (event space) if you eat cake (sample space) is 1.

Provable Identities of Probability

Identity 1:

$$P(E^c) = 1 - P(E) \quad (= P(S) - P(E))$$

Identity 2:

If $E \subseteq F$, then $P(E) \leq P(F)$

Identity 3:

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

General Inclusion-Exclusion Identity:

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{r=1}^n (-1)^{(r+1)} \sum_{i_1 < \dots < i_r} P(E_{i_1} E_{i_2} \dots E_{i_r})$$

That is complicated notation (thanks Ross). You have a bunch of events. The outer sum loops over the possible orders of event subsets. The “-1” term choses if you add or subtract terms with that order. The inner sum makes the power set of that order.

Equally Likely Events

Some sample spaces have equally likely outcomes. We like those sample spaces.

1. Coin flip: $S = \{\text{Head, Tails}\}$
2. Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
3. Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

$$P(\text{Each outcome}) = \frac{1}{|S|}$$

In that case:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

Disclaimer: This handout was made fresh just for you. Did you notice any mistakes? Let Chris know and he will fix them.