Announcements

Sign up for Piazza!  https://piazza.com/stanford/summer2018/cs109
Sign up for Gradescope!  https://gradescope.com/courses/20179 (MWVBP7)
Reference PDFs:  H02 (Calculation reference), H03 (Python installation)
Python tutorial:  Tomorrow (Thursday 6/28)
  2:30-3:30pm, Gates B03
  (will be recorded)

Problem Set 1 is out!  Due: Friday 7/6, 1:30pm (before class)
Participation

Attendance starts today!  tinyurl.com/cs109summer2018

Attend ≥ 80% classes:
  • In-person: enter in-class code by midnight
  • Remote/SCPD: enter remote code within 24hr (by next day 1:30p PT)

Tiny feedback starts today!
  • Two 2-minute forms throughout the quarter
  • Check your Stanford email for assignments

35% grade from final exam:
   Final exam = max(final, 6/7* final + 1/7 * participation)
Goals for today

More Combinatorics

- Permutations of indistinct objects
- Combinations
- Binomial and multinomial coefficients
- Counting integer solutions of equations
Summary from last time

One experiment, multiple outcomes

- Sum Rule of Counting
  (general form) Inclusion-Exclusion Principle

Series of experiments

- Product Rule of Counting
  (general form) General Principle of Counting

General Pigeonhole Principle
Permutations

A permutation is an ordered arrangement of distinct objects.

$n$ distinct objects can be permuted in:

$n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1 = n!$ ways
Bit strings, revisited

Problem:

How many distinct bit strings can be formed from three 0’s and two 1’s?

Solution:

5 total digits: \[
\frac{5!}{3! \cdot 2!} = \frac{120}{6 \cdot 2} = 10
\]

Ways to permute 0’s

Ways to permute 1’s
Permutations of Indistinct Objects

If there are \( n \) objects such that:

- \( n_1 \) are the same (indistinguishable),
- \( n_2 \) are the same,

... and \( n_r \) are the same, where \( \sum_{i=1}^{r} n_i = n \)

Then there are:

\[
\frac{n!}{n_1! \cdot n_2! \cdots n_r!}
\]

distinct permutations of the objects.
Computer maintenance, revisited

Problem:

4 PCs, 3 Macs, 2 Linux computers are to be scheduled for maintenance. How many ways can they be scheduled if:

1. ...they can be arranged in any order? \(9! = 362,880\)

2. ...machines of the same type are indistinguishable?

Solution:

Permute 9 machines:

\[
\frac{9!}{4! \ 3! \ 2!} = 1260
\]

Interchangeable PCs  Macs  Linux
Observations:
• One of the digits was repeated, but we don’t know which
• Once we know the repeated digit, we permute $n = 4$ digits:
  2 indistinguishable digits, the other two distinguishable

Problem:
 iPhones have 4-digit passcodes. How many distinct passcodes are possible if:

1. ...there are 4 smudges over 4 digits? $4! = 24$
2. ...there are 3 smudges over 3 digits?

Solution:

Multi-part experiment!
Phone passcodes, revisited

Problem:
 iPhones have 4-digit passcodes.
How many distinct passcodes are possible if:
1. …there are 4 smudges over 4 digits?  
   \[4! = 24\]
2. …there are 3 smudges over 3 digits?

Solution:
Multi-part experiment (use Product rule):
1. choose a digit to repeat
2. Count the number of outcomes with that repeated digit
   \[3 \cdot \frac{4!}{2!1!1!} = 3 \cdot 12 = 36\]
Phone passcodes, revisited

Problem:

iPhones have 4-digit passcodes. How many distinct passcodes are possible if:

1. ...there are 4 smudges over 4 digits? \[4! = 24\]
2. ...there are 3 smudges over 3 digits? \[3 \cdot 12 = 36\]
3. ...there are 2 smudges over 2 digits?

Solution:

Observations:

• Either we used two digits twice each, OR
• We used one digit 3 times, and the other digit once.

Two types of experiments!
Phone passcodes, revisited

Problem:

iPhones have 4-digit passcodes. How many distinct passcodes are possible if:

1. ...there are 4 smudges over 4 digits?  \(4! = 24\)
2. ...there are 3 smudges over 3 digits?  \(3 \cdot 12 = 36\)
3. ...there are 2 smudges over 2 digits?  \(6 + 8 = 14\)

Solution:

Product Rule within each experiment

\[
\frac{4!}{2!2!} + 2 \cdot \frac{4!}{3!1!} = 6 + 2 \cdot 4 = 14
\]

Sum Rule for each type of experiment

(2 digits used 2x) (use a digit 1x, other one 3x)
Combinations

A *combination* is an unordered selection of *r* objects from a set of *n* objects.

The number of ways of making this selection is:

\[
\frac{n!}{r!(n-r)!} = \binom{n}{r} \text{ ways}
\]

“n choose r”
Combinations

A combination is an unordered selection of \( r \) objects from a set of \( n \) objects.

The number of ways of making this selection is:

\[
\frac{n!}{r!(n-r)!} = \binom{n}{r} \text{ ways} = \binom{n}{n-r} \text{ ways}
\]

“\( n \) choose \( r \)”

Multipart experiment:

1. Permute all \( n \) objects.
2. Select the first \( r \) objects in the permutation.
3. Order of first \( r \) objects
4. Order of unselected \( (n-r) \) objects

\[
\frac{n! \cdot 1}{r!(n-r)!}
\]
Video games

Problem:
We have 5 video games.

1. How many ways to select 3 games from 6?

Solution:

\[
\binom{6}{3} = \frac{6!}{3! \ 3!} = 20
\]
Video games

Problem:
We want to select 3 games from a set of 6 games.

2. How about if we don’t want both Pokemon Moon games together?

Solution 1:

3 cases, aka 3 types of experiments:
• Select Pokemon Moon and 2 other games
  \[1 \cdot \binom{4}{2}\]
• Select Pokemon Ultra Moon and 2 other games
  \[1 \cdot \binom{4}{2}\]
• Select 3 from non-{Moon, Ultra Moon}
  \[\binom{4}{3}\]

Then use Sum Rule to combine.
\[\binom{4}{2} + \binom{4}{2} + \binom{4}{3} = 16\]
Video games, part 2

Problem:
2. How about if we don’t want both Pokemon games together?

Solution 2:

1 normal experiment:
• Select 3 games normally. \( \binom{6}{3} = 20 \)

Identify the forbidden combination.
• Select both Pokemon Moon and Pokemon Ultra Moon and 1 other game. \( 1 \cdot 1 \cdot \binom{4}{1} = 4 \)

Subtract. \( 20 - 4 = 16 \)
Combinations, recursively

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]
Combinations, recursively

\( \binom{n}{r} = \binom{12}{5} \)

n = 12 points
Choose r = 5 points
Combinations, recursively

From the perspective of a single point....

One possibility

Another possibility

That point can be included...

...Or that point can be excluded.

Sum Rule of Counting

(number of solutions including this point)

+ 

(number of solutions excluding this point)
Combinations, recursively

\[
\binom{n}{r} = 1 \cdot \binom{n-1}{r-1} + \binom{n-1}{r}
\]

Sum Rule of Counting

1 point

n-1 other points

(number of solutions including this point)

+ (number of solutions excluding this point)
Combinations as a function

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

\[
\binom{n-1}{r-1} + \binom{n-1}{r}
\]

def C_iter(n, r):
    num = fact(n)
    denom = fact(r) * fact(n-r)
    return num / denom

def C_recur(n, r):
    if r == 0 or n == r:
        return 1
    return C_recur(n-1, r-1) + C_recur(n-1, r)

fact(n) returns the factorial of n, i.e. n!
Break!

Attendance: tinyurl.com/cs109summer2018
Binomial coefficients

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

“n choose r”

Why the name?

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{r} x^k y^{n-k}
\]

The Binomial Theorem
Multinomial coefficients

\[
\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! \, n_2! \cdots n_r!}
\]

where \( \sum_{i=1}^{r} n_i = n \)

Two interpretations:

1. Permutations of indistinct objects
2. Number of ways to divide \( n \) distinct objects into \( r \) groups of \( n_1, n_2, \ldots, n_r \) items respectively
Datacenter computers

Problem:
13 different computers are to be allocated to 3 datacenters as follows:
How many different divisions are possible?

Solution:

Direct application of multinomial coefficients:

\[
\binom{13}{6,4,3} = \frac{13!}{6!4!3!} = 60,060
\]
Datacenter computers

Problem:
13 different computers are to be allocated to 3 datacenters as follows:

How many different divisions are possible?

Solution:

Multipart experiment, Product Rule:

\[
\binom{13}{6} \binom{7}{4} \binom{3}{3} = \frac{13!}{6!7!} \frac{7!}{4!3!} \frac{3!}{3!0!} = \frac{13!}{6!4!3!} = \binom{13}{6,4,3}
\]

\[
= 60,060
\]
Balls and urns

n balls

r urns

Literally stop right there...
Distinguishable

Balls and urns Strings and buckets

\( n \) balls strings

\( r \) urns buckets

\( r^n \) ways
Indistinguishable strings and buckets

Indistinguishable!

n strings

r buckets

asdfasdfasdfasdfs asdfasdfasdfasdfs asdfasdfasdfasdfs asdfasdfasdfasdfs
Divider Method

The number of ways to place $n$ indistinguishable items into $r$ containers is equivalent to the number of ways to permute $n+r-1$ objects such that:

- $n$ items are the same
- $r-1$ dividers are the same

Total ways:

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{r-1}$$
Problem:
How many integer solutions are there to the following equation:

\[ x_1 + x_2 + \ldots + x_r = n, \]

where for all \( i \), \( 0 \leq x_i \leq n \)?

Solution:

\[ \binom{n + r - 1}{r - 1} \]

\( n \) indistinguishable balls, \( r \) urns
Integer solutions to equations

Problem:
How many integer solutions are there to the following equation:

\[ x_1 + x_2 + \ldots + x_r = n, \]

where for all \( i \), \( x_i \) is positive? \( 1 \leq x_i \leq n \)

Solution:

\[
\begin{array}{|c|c|c|}
\hline
1+? & 1+? & \ldots & 1+? \\
1+? & 1+? & \ldots & 1+? \\
\hline
\end{array}
\]

\[
x[1] \quad x[2] \quad \ldots \quad x[r]
\]

\[
\binom{n-r+r-1}{r-1} = \binom{n-1}{r-1}
\]
Venture Capitalists

Problem:
You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways do you have to allocate your $10 million?

Solution:
\[ x_1 + x_2 + x_3 + x_4 = 10, \text{ where } x_i \geq 0 \]

10 balls, 4 urns
\[ \binom{10 + 4 - 1}{4 - 1} = \binom{13}{3} = 286 \]
Venture Capitalists

Problem:
You have $10 million to invest in 4 companies (in $1 million increments).

2. What if you don’t have to invest all of your money?

Solution:
Your bank account

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 10, \text{ where } x_i \geq 0 \]

10 balls, 5 urns

\[
\binom{10 + 5 - 1}{5 - 1} = \binom{14}{4} = 1001
\]
Venture Capitalists

Problem:
You have $10 million to invest in 4 companies (in $1 million increments).

3. What if you want to invest at least $3 million in company 1 (and you invest all $10 million)?

Solution:

\[ x_1 + x_2 + x_3 + x_4 = 7, \text{ where } x_i \geq 0 \]

7 balls, 4 urns
\[
\binom{7 + 4 - 1}{4 - 1} = \binom{10}{3} = 120
\]
Summary of Combinatorics

Counting operations on $n$ objects

- **Ordered permutations**
  - Distinguishable: $n!$
  - Some indistinguishable:
    $$\frac{n!}{n_1!n_2! \ldots}$$

- **Choose $k$ combinations**
  - Distinguishable: $\binom{n}{k}$
  - Indistinguishable:
    $$\frac{n}{n_1, n_2, \ldots, n_r}$$

- **Put in $r$ buckets**
  - Distinguishable: $r^n$
  - Indistinguishable:
    $$\frac{(n + r - 1)!}{n!(r - 1)!}$$
Mo’ Practice, Mo’ Fun

Problem:
You want to pet 3 dogs, 3 cats, and 2 hamsters, but you can only pet the animals one at a time.

1. How many ways are there to pet your distinct animal friends?
2. What if you can’t distinguish between any of the dogs?

Solution:
1. Distinct, ordered \((3+3+2)! = 8! = 40,320\)
2. Some indistinct, ordered \(\frac{8!}{3!} = 6,720\)
# Mo’ Practice, Mo’ Fun

**Problem:**
You want to pet 3 dogs, 3 cats, and 2 hamsters, but you can only pet the animals one at a time.

3. How many ways can you pet your animal friends if you always want to pet a dog after a cat?

**Solution:**
Distinct, ordered

\[(5!) \cdot 3! \cdot 1 = 720\]

<table>
<thead>
<tr>
<th>Multi-part experiment!</th>
<th>Group line (hamster and dog only)</th>
<th>5!</th>
<th>3!</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cat line</td>
<td>Insert cats</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Mo’ Practice, Mo’ Fun

Problem:
At a zoo, there are 10 different species of birds that need to be assigned to 3 different aviaries. Each aviary must have at least 2 species, and no species can be in more than one aviary.

How many ways are there to assign the 10 species to the 3 aviaries?

Solution:
Distinct buckets + combinations

\[
\binom{10}{2,2,2,4} \times 3^4
\]

Multi-part experiment!

Choose the 2 species in each aviary

Bucket the remaining species

1,530,900