02: Combinatorics

Lisa Yan and Jerry Cain
September 16, 2020
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Today’s discussion thread: [https://us.edstem.org/courses/2678/discussion/124109](https://us.edstem.org/courses/2678/discussion/124109)
Permutations II
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
- Choose $k$ objects (combinations)
- Put objects in $r$ buckets

Distinct (distinguishable)
Sort $n$ distinct objects

Ayesha  Tim  Irina  Joey  Waddie

# of permutations =
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - $n!$

- Choose $k$ objects (combinations)
  - Some distinct

- Put objects in $r$ buckets
Sort semi-distinct objects

All distinct

Ayesha  Tim  Irina  Joey  Waddie

Some indistinct

Coke  Tim  Coke  Joey  Waddie

Order $n$ distinct objects  $n!$
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \text{permutations considering some objects are indistinct} \times \text{Permutations of just the indistinct objects}
\]
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \text{permutations considering some objects are indistinct}
\]

\[
\frac{\text{Permutations of just the indistinct objects}}{\text{Permutations of just the indistinct objects}}
\]
General approach to counting permutations

When there are $n$ objects such that
- $n_1$ are the same (indistinguishable or indistinct), and
- $n_2$ are the same, and
- ...
- $n_r$ are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1! n_2! \cdots n_r!}.$$ 

For each group of indistinct objects,
Divide by the overcounted permutations.
Sort semi-distinct objects

How many permutations?

Coke  Coke0  Coke  Coke0  Coke0
Summary of Combinatorics

Counting tasks on \( n \) objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - \( n! \)
  - Some distinct
    - \( \frac{n!}{n_1! n_2! \cdots n_r!} \)
- Choose \( k \) objects (combinations)
  - \( \frac{n!}{n_1! n_2! \cdots n_r!} \)
- Put objects in \( r \) buckets
Strings

How many orderings of letters are possible for the following strings?

1. **BOBA**

2. **MISSISSIPPI**

Order $n$ semi-distinct objects $\frac{n!}{n_1!n_2!...n_r!}$
How many orderings of letters are possible for the following strings?

1. BOBA
   \[
   = \frac{4!}{2!} = 12
   \]

2. MISSISSIPPI
   \[
   = \frac{11!}{1!4!4!2!} = 34,650
   \]
How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers?

Total = 6!

= 720 passcodes
Unique 6-digit passcodes with five smudges

How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

Steps:
1. Choose digit to repeat 5 outcomes
2. Create passcode (sort 6 digits: 4 distinct, 2 indistinct)

Total = 5 × $\frac{6!}{2!}$

= 1,800 passcodes
Combinations I
Summary of Combinatorics

Counting tasks on \( n \) objects

- **Sort objects (permutations)**
  - Distinct (distinguishable)
  - \( n! \)

- **Choose \( k \) objects (combinations)**
  - Some distinct
  - \( \frac{n!}{n_1! n_2! \cdots n_r!} \)

- **Put objects in \( r \) buckets**
  - Distinct
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...
Combinations with cake

There are \( n = 20 \) people.
How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line

\( n! \) ways
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line $n!$ ways
2. Put first $k$ in cake room 1 way
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   - $n!$ ways

2. Put first $k$ in cake room
   - 1 way
Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line $n!$ ways
2. Put first $k$ in cake room 1 way
3. Allow cake group to mingle

$k!$ different permutations lead to the same mingle

$n!$ ways
Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line  
   $n!$ ways

2. Put first $k$ in cake room  
   1 way

3. Allow cake group to mingle  
   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

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Combinations with cake

There are \( n = 20 \) people.
How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line
   \( n! \) ways

2. Put first \( k \) in cake room
   1 way

3. Allow cake group to mingle
   \( k! \) different permutations lead to the same mingle

4. Allow non-cake group to mingle
   \( (n - k)! \) different permutations lead to the same mingle
Combinations

A combination is an unordered selection of $k$ objects from a set of $n$ distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k! (n - k)!} = n! \times \frac{1}{k!} \times \frac{1}{(n - k)!}$$

1. Order $n$ distinct objects
2. Take first $k$ as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice
Combinations

A combination is an unordered selection of $k$ objects from a set of $n$ distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k! (n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

Note: $\binom{n}{n-k} = \binom{n}{k}$
Probability textbooks

How many ways are there to choose 3 books from a set of 6 distinct books?

\[
\binom{6}{3} = \frac{6!}{3! 3!} = 20 \text{ ways}
\]
Combinations II
Summary of Combinatorics

Counting tasks on $n$ objects

- **Sort objects** (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1!n_2!\cdots n_r!}$

- **Choose $k$ objects** (combinations)
  - Distinct
    - 1 group
    - $\binom{n}{k}$
  - Some distinct
    - $r$ groups

- **Put objects in $r$ buckets**
General approach to combinations

The number of ways to choose $r$ groups of $n$ distinct objects such that

For all $i = 1, \ldots, r$, group $i$ has size $n_i$, and

$$\sum_{i=1}^{r} n_i = n \text{ (all objects are assigned), is}$$

$$\frac{n!}{n_1! n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}$$

Multinomial coefficient
13 different computers are to be allocated to 3 datacenters as shown in the table:

<table>
<thead>
<tr>
<th>Datacenter</th>
<th># machines</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
</tbody>
</table>

How many different divisions are possible?

A. \( \binom{13}{6,4,3} = 60,060 \)
B. \( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)
C. \( 6 \cdot 1001 \cdot 10 = 60,060 \)
D. A and B
E. All of the above

Choose \( k \) of \( n \) distinct objects into \( r \) groups of size \( n_1, \ldots, n_r \):

\( \binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!} \)
Datacenters

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Datacenters

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How many different divisions are possible?

A. \( \binom{13}{6,4,3} = 60,060 \)

Strategy: Combinations into 3 groups

Group 1 (datacenter A): \( n_1 = 6 \)

Group 2 (datacenter B): \( n_2 = 4 \)

Group 3 (datacenter C): \( n_3 = 3 \)
Datacenters

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

A. \( \binom{13}{6,4,3} = 60,060 \)

Strategy: Combinations into 3 groups

Group 1 (datacenter A): \( n_1 = 6 \)
Group 2 (datacenter B): \( n_2 = 4 \)
Group 3 (datacenter C): \( n_3 = 3 \)

B. \( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)

Strategy: Product rule with 3 steps

1. Choose 6 computers for A \( \binom{13}{6} \)
2. Choose 4 computers for B \( \binom{7}{4} \)
3. Choose 3 computers for C \( \binom{3}{3} \)
13 different computers are to be allocated to 3 datacenters as shown in the table:

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B. \( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)

Strategy: Product rule with 3 steps

1. Choose 6 computers for A \( \binom{13}{6} \)
2. Choose 4 computers for B \( \binom{7}{4} \)
3. Choose 3 computers for C \( \binom{3}{3} \)

Your approach will determine if you use binomial/multinomial coefficients or factorials.
02: Combinatorics (live)

Lisa Yan
September 16, 2020
Reminders: Lecture with zoom

- Turn on your camera if you are able, mute your mic in the big room
- Virtual backgrounds are encouraged (classroom-appropriate)

Today’s discussion thread: https://us.edstem.org/courses/2678/discussion/124109
Summary of Combinatorics

Counting tasks on \( n \) objects

- **Sort objects** (permutations)
  - Distinct (distinguishable)
  - \( n! \)

- **Choose \( k \) objects** (combinations)
  - Some distinct
  - \( \frac{n!}{n_1! n_2! \cdots n_r!} \)
  - Distinct
  - \( n \choose k \)

- **Put objects in \( r \) buckets**
  - 1 group
  - \( n \choose n_1, n_2, \ldots, n_r \)
  - \( r \) groups
Summary of Combinatorics

Counting tasks on $n$ objects

- **Sort objects (permutations)**
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1! n_2! \cdots n_r!}$

- **Choose $k$ objects (combinations)**
  - Distinct
    - $\binom{n}{k}$
  - Some distinct
    - $\binom{n}{n_1, n_2, \ldots, n_r}$
  - Indistinct?

- **Put objects in $r$ buckets**
  - $\binom{n}{n_1, n_2, \ldots, n_r}$
Think

Slide 42 is a question to think over by yourself (~1min).

Post any clarifications here or in chat!

https://us.edstem.org/courses/2678/discussion/124109
A trick question

How many distinct (distinguishable) ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?

A. \( \binom{6}{1,2,3} \)
B. \( \frac{6!}{1!2!3!} \)
C. 0
D. 1
E. Both A and B
F. Something else

(by yourself)
A trick question

How many **distinct** (distinguishable) ways are there to group 6 **indistinct** (indistinguishable) objects into 3 groups, where groups A, B, and C have sizes 1, 2, and 3, respectively?

A. \( \binom{6}{1,2,3} \)

B. \( \frac{6!}{1!2!3!} \)

C. 0

D. 1

E. Both A and B

F. Something else
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?
Slide 46 is a question to think over by yourself (~2min).

Post any clarifications here or in chat!

https://us.edstem.org/courses/2678/discussion/124109
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[ \binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways} \]

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. \( \binom{6}{3} - \binom{6}{2} = 5 \text{ ways} \)
B. \( \frac{6!}{3!3!2!} = 10 \)
C. \( 2 \cdot \binom{4}{2} + \binom{4}{3} = 16 \)
D. \( \binom{6}{3} - \binom{4}{1} = 16 \)
E. Both C and D
F. Something else

Ask: [https://us.edstem.org/courses/2678/discussion/124109](https://us.edstem.org/courses/2678/discussion/124109)
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[
\binom{6}{3} = \frac{6!}{3! \cdot 3!} = 20 \text{ ways}
\]

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 1: Sum Rule
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[
\binom{6}{3} = \frac{6!}{3! 3!} = 20 \text{ ways}
\]

2. What if we do not want to read both the 9th and 10th edition of Ross?

Strategy 2: “Forbidden method” (unofficial name)

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.
Interlude for jokes/announcements
Announcements

Problem Set #1
Out: today
Due: Friday 9/25, 1:00pm
Covers: through Friday 9/18

Python tutorial (2 timeslots)
When: Friday 12:00-1:00am PT
      Friday 2:00-3:00pm PT
Recorded?: yes
Notes: to be posted online

Section sign-ups/Acquaintance form
Form released: later today
Form due: Saturday 5:00pm 9/19
Results: latest Sunday

Getting help
Ed discussion: find study buddies!
Office hours: start today
https://web.stanford.edu/class/cs109/stanford/staff.html
Buckets and The Divider Method
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$

- Choose $k$ objects (combinations)
  - Distinct
    - $\binom{n}{k}$
  - Some distinct
    - $\binom{n}{n_1, n_2, \ldots, n_r}$

- Put objects in $r$ buckets
  - Distinct
  - Indistinct
Balls and urns \hspace{1cm} Hash tables and distinct strings

How many ways are there to hash $n$ distinct strings to $r$ buckets?

Steps:
1. Bucket 1\textsuperscript{st} string
2. Bucket 2\textsuperscript{nd} string
   
   ... 

$n$. Bucket $n$\textsuperscript{th} string
Summary of Combinatorics

Counting tasks on $n$ objects

- **Sort objects** (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1! n_2! \cdots n_r!}$

- **Choose $k$ objects** (combinations)
  - Distinct
    - $\binom{n}{k}$
  - Some distinct
    - $\binom{n}{n_1, n_2, \ldots, n_r}$

- **Put objects in $r$ buckets**
  - Distinct
    - $r^n$
  - Indistinct

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Servers and indistinct requests

How many ways are there to distribute \( n \) indistinct web requests to \( r \) servers?

Goal
Server 1 has \( x_1 \) requests,
Server 2 has \( x_2 \) requests,
...
Server \( r \) has \( x_r \) requests (the rest)
Simple example: \( n = 3 \) requests and \( r = 2 \) servers
Bicycle helmet sales

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?
Bicycle helmet sales

1 possible assignment outcome:

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

Consider the following generative process...
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

Goal: Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct
The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

**Goal** Order $n$ indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

1. Order $n$ distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\[
(n + r - 1)!
\]

2. Make \( n \) objects indistinct

\[
\frac{1}{n!}
\]
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

**Goal**: Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\[
(n + r - 1)!
\]

2. Make \( n \) objects indistinct

\[
\frac{1}{n!}
\]

3. Make \( r - 1 \) dividers indistinct

\[
\frac{1}{(r - 1)!}
\]
The divider method

The number of ways to distribute $n$ indistinct objects into $r$ buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that $n$ are indistinct objects, and $r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$
Integer solutions to equations

How many integer solutions are there to the following equation:

\[ x_1 + x_2 + \cdots + x_r = n, \]

where for all \( i \), \( x_i \) is an integer such that \( 0 \leq x_i \leq n \)?

Positive integer equations can be solved with the divider method.

Divider method \( \binom{n + r - 1}{r - 1} \) (\( n \) indistinct objects, \( r \) buckets)
Lecture with **zoom**

Breakout Rooms for working through lecture exercises

- We may incorporate some of these during lecture
- You are *always welcome* to exit breakout rooms if you are more comfortable staying in the main room
Introduce yourself!

Then check out the three questions on the next slide (Slide 67).
Post any clarifications here:

https://us.edstem.org/courses/2678/discussion/124109

Breakout Room time: 5 minutes

We’ll then all come back as a big group to go over our approach.
You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don’t have to invest all your money?
You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?

Set up

\[ x_1 + x_2 + x_3 + x_4 = 10 \]

\[ x_i: \text{amount invested in company } i \]

\[ x_i \geq 0 \]
You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?

Set up

\[ x_1 + x_2 + x_3 + x_4 = 10 \]

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You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don’t have to invest all your money?

Set up

\[ x_1 + x_2 + x_3 + x_4 \leq 10 \]

\[ x_i: \text{amount invested in company } i \]

\[ x_i \geq 0 \]
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
- Choose $k$ objects (combinations)
  - Some distinct
  - Distinct
  - 1 group
  - $r$ groups
- Put objects in $r$ buckets
  - Distinct
  - Indistinct

- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases
See you next time...