03: Intro to Probability

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Summary of Combinatorics

Counting tasks on $n$ objects

- **Sort objects (permutations)**
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1!n_2!\cdots n_r!}$

- **Choose $k$ objects (combinations)**
  - Distinct
    - $\binom{n}{k}$
  - Some distinct
    - $\left( \begin{array}{c} n \\ n_1, n_2, \ldots, n_r \end{array} \right)$

- **Put objects in $r$ buckets**
  - Distinct
    - $r^n$
  - Indistinct
    - $\frac{(n + r - 1)!}{n!(r - 1)!}$
Summary of Combinatorics

Counting tasks on \( n \) objects

Sort objects (permutations)

Choose \( k \) objects (combinations)

Put objects in \( r \) buckets

Distinct (distinguishable)

Some distinct

1 group

Distinct

\( k \) groups

Distinct

Indistinct

Review

• Determine if objects are distinct
• Use Product Rule if several steps
• Use Inclusion-Exclusion if different cases

Lisa Yan, CS109, 2019
For a DNA tree, we need to calculate the DNA distance between each pair of animals. How many calculations are needed, i.e., how many distinct pairs of $n$ animals are there?
DNA distance

For a DNA tree, we need to calculate the DNA distance between each pair of animals. How many calculations are needed, i.e., how many distinct pairs of $n$ animals are there?

Approach 1:

$$\frac{n^2 - n}{2}$$

Approach 2:

$$\binom{n}{2}$$
Today’s plan

Key definitions: sample spaces and events

Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability
Key definitions

An experiment in probability:

Sample Space, $S$: The set of all possible outcomes of an experiment.

Event, $E$: Some subset of $S$ ($E \subseteq S$).
Key definitions

Sample Space, $S$
- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, \ x \geq 0\}$
- YouTube hours in a day
  $S = \{x \mid x \in \mathbb{R}, \ 0 \leq x \leq 24\}$

Event, $E$
- Flip lands heads
  $E = \{\text{Heads}\}$
- $\geq 1$ head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day ($\leq 20$ emails)
  $E = \{x \mid x \in \mathbb{Z}, \ 0 \leq x \leq 20\}$
- Wasted day ($\geq 5$ YT hours):
  $E = \{x \mid x \in \mathbb{R}, \ 5 \leq x \leq 24\}$
What is a probability?

A number between 0 and 1 to which we ascribe meaning.*

*our belief that an event $E$ occurs.
What is a probability?

Let $E$ = the set of outcomes where you hit the target.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$n = \# \text{ of total trials}$

$n(E) = \# \text{ trials where } E \text{ occurs}$
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- \( n \) = # of total trials
- \( n(E) \) = # trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.00 \]
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.50 \]

\( n \) = # of total trials

\( n(E) \) = # trials where \( E \) occurs
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- \( n \) = # of total trials
- \( n(E) \) = # trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) = 0.66 \]
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- \( n \) = \# of total trials
- \( n(E) \) = \# trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.46 \]
Not just yet...
Today’s plan

Key definitions: sample spaces and events

Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
- Dice roll
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - Let $E = \{1, 2\}$, and $F = \{2, 3\}$
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
Dice roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def** Union of events, $E \cup F$

The event containing all outcomes in $E$ or $F$.

$E \cup F = \{1, 2, 3\}$
Quick review of sets

Def Intersection of events, $E \cap F$

The event containing all outcomes in $E$ and $F$.

Def Mutually exclusive events $F$ and $G$ means that $F \cap G = \emptyset$

$E$ and $F$ are events in $S$.

Experiment:

Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

$E \cap F = EF = \{2\}$
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
Dice roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2, 3\}$

\[ E^C = \{3, 4, 5, 6\} \]

**def** Complement of event $E$, $E^C$

The event containing all outcomes in that are *not* in $E$. 
3 Axioms of Probability

Definition of probability: \[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

Axiom 1: \[ 0 \leq P(E) \leq 1 \]

Axiom 2: \[ P(S) = 1 \]

Axiom 3: If \( E \) and \( F \) are mutually exclusive (\( E \cap F = \emptyset \)), then \[ P(E \cup F) = P(E) + P(F) \]
Axiom 3 is the (analytically) useful Axiom

**Axiom 3:** If $E$ and $F$ are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i)$$

(like the Sum Rule of Counting, but for probabilities)
Today’s plan

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Corollaries of Axioms of Probability
Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

- Coin flip: $S = \{\text{Head, Tails}\}$
- Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
- Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

P(Each outcome) $= \frac{1}{|S|}$

In that case, $P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{|E|}{|S|}$ (by Axiom 3)
Roll two dice

Roll two 6-sided dice. What is $P(\text{sum} = 7)$?

$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

Equally likely outcomes
Target revisited
Target revisited

Let $E$ = the set of outcomes where you hit the target.

The dart is equally likely to land anywhere on the screen.

What is $P(E)$, the probability of hitting the target?

Screen size = $800 \times 800$  \[|S| = 800^2\]

Radius of target: 200  \[|E| = \pi \cdot 200^2\]

\[P(E) = \frac{|E|}{|S|} = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963\]
Target revisited

Let $E =$ the set of outcomes where you hit the target.

The dart is equally likely to land anywhere on the screen.

What is $P(E)$, the probability of hitting the target?

Screen size = $800 \times 800$ \quad $|S| = 800^2$

Radius of target: 200 \quad $|E| = \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$
Not equally likely outcomes

Play the lottery.
What is $P(\text{win})$?

$S = \{\text{Lose, Win}\}$

$E = \{\text{Win}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%?$$

The hard part: defining equally likely outcomes consistently across sample space and events
Cats and carrots

4 cats and 3 carrots in a bag. 3 drawn.
What is $P(1$ cat and $2$ carrots drawn)\

\[ P(E) = \frac{|E|}{|S|} \text{ Equally likely outcomes} \]

- A. $\frac{3}{7}$
- B. $\frac{1}{4} \cdot \frac{2}{3}$
- C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$
- D. $\frac{12}{35}$
- E. Zero/other

Note: Do indistinct objects give you an equally likely sample space?
Cats and carrots

4 cats and 3 carrots in a bag. 3 drawn.
What is \( P(1 \text{ cat and 2 carrots drawn}) \)?

**Note**: Do indistinct objects give you an equally likely sample space?

\[
P(E) = \frac{|E|}{|S|} \quad \text{Equally likely outcomes}
\]

A. \( \frac{3}{7} \)

B. \( \frac{1}{4} \cdot \frac{2}{3} \)

C. \( \frac{4}{7} + 2 \cdot \frac{3}{6} \)

D. \( \frac{12}{35} \)

E. Zero/other

Make indistinct items distinct to get equally likely outcomes.
Cats and carrots

4 cats and 3 carrots in a bag. 3 drawn. What is P(1 cat and 2 carrots drawn)?

Define

- $S = \text{Pick 3 distinct items}$
  
  \[ |S| = 7 \cdot 6 \cdot 5 = 210 \]

- $E = \text{1 distinct cat, 2 distinct carrots}$
  
  Pick Cat $1^{\text{st}}, 2^{\text{nd}}, \text{or} 3^{\text{rd}}$

  \[ |E| = 4 \cdot 3 \cdot 2 + 3 \cdot 4 \cdot 2 + 3 \cdot 2 \cdot 4 = 72 \]

Compute

\[ P(E) = \frac{72}{210} = \frac{12}{35} \]
Cats and carrots

4 cats and 3 carrots in a bag. 3 drawn. What is \( P(1 \text{ cat and 2 carrots drawn}) \)?

Define

- \( S \) = Pick 3 distinct items
- \( E \) = 1 distinct cat, 2 distinct carrots

Ordered

\[
|S| = 7 \cdot 6 \cdot 5 = 210
\]

Unordered

\[
|S| = \binom{7}{3}
\]

Pick Cat 1\(^{\text{st}}\), 2\(^{\text{nd}}\), or 3\(^{\text{rd}}\)

\[
|E| = 4 \cdot 3 \cdot 2 + 3 \cdot 4 \cdot 2 + 3 \cdot 2 \cdot 4 = 72
\]

\[
P(E) = \frac{72}{210} = \frac{12}{35}
\]

\[
P(E) = \frac{12}{35}
\]
Break for Friday/announcements
Announcements

Section sign-ups
Preference form: out
Due: Saturday 9/28
Results: latest Monday

Concept check
Due: Tuesday 1:00pm

Python tutorial
When: Friday 3:30-4:20pm
Location: Hewlett 102
Recorded?: Yes!
Notes: to be posted online
Installation: On Piazza
Any Poker Straight

Consider 5-card poker hands.
• “straight” is 5 consecutive rank cards of any suit

What is $P(\text{Poker straight})$?

- What is an example of an outcome?
- Is each outcome equally likely?
- Should objects be ordered or unordered?
Any Poker Straight

Consider 5-card poker hands.
- “straight” is 5 consecutive rank cards of any suit

What is $P(\text{Poker straight})$?

Define
- $S$ (unordered) $|S| = \binom{52}{5}$
- $E$ (unordered, consistent with $S$) $|E| = 10 \cdot \binom{4}{1}^5$

Compute $P(\text{Poker straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$
“Official” Poker Straight

Consider 5-card poker hands.

• “straight” is 5 consecutive rank cards of any suit
• “straight flush” is 5 consecutive rank cards of \textit{same} suit

What is \(P(\text{Poker straight, but not straight flush})\)?

Define

\( S \) (unordered) \quad |S| = \binom{52}{5}

\( E \) (unordered, consistent with \( S \)) \quad |E| = 10 \cdot \binom{4}{1}^5 - 10 \cdot \binom{4}{1}

Compute \quad P(\text{Official Poker straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5 - 10 \cdot \binom{4}{1}}{\binom{52}{5}} \approx 0.00392
Chip defect detection

$n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing.

What is $P($defective chip is in $k$ selected chips$)?$ 

Define

- $S$ (unordered) \[ |S| = \binom{n}{k} \]
- $E$ (unordered, consistent with $S$) \[ |E| = \binom{1}{1} \binom{n-1}{k-1} \]

Compute \[ P(E) = \frac{(n-1)}{k-1} \binom{n}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)! k!}{n! (k-1)!} = \frac{k}{n} \]
Chip defect detection, solution #2

$n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing.

What is $P$(defective chip is in $k$ selected chips?)

Redefine experiment

1. Choose $k$ indistinct chips (1 way)
2. Throw a dart and make one defective

Define

- $S$ (unordered) $|S| = 1 \cdot n$
- $E$ (unordered, consistent with $S$) $|E| = 1 \cdot k$

$$P(E) = \frac{k}{n}$$
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Corollaries of Axioms of Probability
Axioms of Probability

Definition of probability: \( P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \)

**Axiom 1:** \( 0 \leq P(E) \leq 1 \)

**Axiom 2:** \( P(S) = 1 \)

**Axiom 3:** If \( E \) and \( F \) are mutually exclusive (\( E \cap F = \emptyset \)), then \( P(E \cup F) = P(E) + P(F) \)
3 Corollaries of Axioms of Probability

Corollary 1: \[ P(E^C) = 1 - P(E) \]
Proof of Corollary 1

Corollary 1: \( P(E^C) = 1 - P(E) \)

Proof:

\( E, E^C \) are mutually exclusive

Definition of \( E^C \)

\[ P(E \cup E^C) = P(E) + P(E^C) \]

Axiom 3

\[ S = E \cup E^C \]

Everything must either be in \( E \) or \( E^C \), by definition

\[ 1 = P(S) = P(E) + P(E^C) \]

Axiom 2

\[ P(E^C) = 1 - P(E) \]

Rearrange
3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$
(Inclusion-Exclusion Principle for Probability)
Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: \[ P(E \cup F) = P(E) + P(F) - P(EF) \]
(Inclusion-Exclusion Principle for Probability)

General form:
\[ P \left( \bigcup_{i=1}^{n} E_i \right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P \left( \bigcap_{j=1}^{r} E_{i_j} \right) \]

\[ P(E \cup F \cup G) = \]
\[ r = 1: \quad P(E) + P(F) + P(G) \]
\[ r = 2: \quad - P(E \cap F) - P(E \cap G) - P(F \cap G) \]
\[ r = 3: \quad + P(E \cap F \cap G) \]
Takeaway: Mutually exclusive events

Axiom 3, Mutually exclusive events

\[ P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i) \]

Inclusion-Exclusion Principle

\[ P \left( \bigcup_{i=1}^{n} E_i \right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_{1}<...<i_{r}} P \left( \bigcap_{j=1}^{r} E_{i_{j}} \right) \]

Design your experiment to compute easier probabilities.
Serendipity

Let it find you.

SERENDIPITY
the effect by which one accidentally stumbles upon something truly wonderful, especially while looking for something entirely unrelated.

WHEN YOU MEET YOUR BEST FRIEND
Somewhere you didn't expect to.
Serendipity

• The population of Stanford is \( n = 17,000 \) people.
• You are friends with \( r = ? \) people.
• Walk into a room, see \( k = 268 \) random people.
• Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know?

Define

• \( S \) (unordered)
  \[ |S| = \binom{n}{k} = \binom{17000}{268} \]

• \( E \): see \( \geq 1 \) friend in the room

How should we compute \( P(E) \)?

A. \( P(\text{exactly 1}) + P(\text{exactly 2}) + P(\text{exactly 3}) + \ldots \)

B. \( 1 - P(\text{see no friends}) \)

👉 It is often much easier to compute \( P(E^c) \).
The Birthday Paradox Problem

What is the probability that in a set of $n$ people, at least one pair of them will share the same birthday?

For you to think about (and discuss in section!)