03: Intro to Probability

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Today’s discussion thread: https://us.edstem.org/courses/2678/discussion/124598
Defining Probability

Gradescope quiz, blank slide deck, etc.
http://cs109.stanford.edu/
Key definitions

An experiment in probability:

Sample Space, $S$: The set of all possible outcomes of an experiment.
Event, $E$: Some subset of $S$ ($E \subseteq S$).
Key definitions

Sample Space, \( S \)

- Coin flip
  \( S = \{\text{Heads, Tails}\} \)

- Flipping two coins
  \( S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\} \)

- Roll of 6-sided die
  \( S = \{1, 2, 3, 4, 5, 6\} \)

- # emails in a day
  \( S = \{x \mid x \in \mathbb{Z}, \ x \geq 0\} \)

- TikTok hours in a day
  \( S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\} \)

Event, \( E \)

- Flip lands heads
  \( E = \{\text{Heads}\} \)

- \( \geq 1 \) head on 2 coin flips
  \( E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\} \)

- Roll is 3 or less:
  \( E = \{1, 2, 3\} \)

- Low email day (\( \leq 20 \) emails)
  \( E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 20\} \)

- Wasted day (\( \geq 5 \) TT hours):
  \( E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\} \)
What is a probability?

A number between 0 and 1 to which we ascribe meaning.*

*our belief that an event \( E \) occurs.
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- \( n \) = \# of total trials
- \( n(E) \) = \# trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

Hit: 0
Thrown: 0

\[ P(E) \approx \]
What is a probability?

Let $E = \{\text{the set of outcomes where you hit the target}\}$.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$n = \# \text{ of total trials}$

$n(E) = \# \text{ trials where } E \text{ occurs}$

$P(E) \approx 0.00$
What is a probability?

\[
P(E) = \lim_{{n \to \infty}} \frac{n(E)}{n}
\]

\(n = \#\) of total trials
\(n(E) = \#\) trials where \(E\) occurs

Let \(E\) = the set of outcomes where you hit the target.

\(P(E) \approx 0.500\)
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

\[ n = \# \text{ of total trials} \]
\[ n(E) = \# \text{ trials where } E \text{ occurs} \]

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.667 \]
What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$n = \# \text{ of total trials}$

$n(E) = \# \text{ trials where } E \text{ occurs}$

Let $E$ = the set of outcomes where you hit the target.

Hit: 11
Thrown: 24

$$P(E) \approx 0.458$$
Not just yet...
Axioms of Probability
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2, 3\}$
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
- Die roll
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def** Union of events, $E \cup F$

The event containing all outcomes in $E$ or $F$.

$$E \cup F = \{1, 2, 3\}$$
Quick review of sets

Review of Sets

**E** and **F** are events in **S**.

Experiment:
Die roll

\[ S = \{1, 2, 3, 4, 5, 6\} \]

Let \( E = \{1, 2\}, \) and \( F = \{2, 3\} \)

**def** Intersection of events, \( E \cap F \)

The event containing all outcomes in **E** and **F**.

\[ E \cap F = EF = \{2\} \]

**def** Mutually exclusive events **F** and **G** means that \( F \cap G = \emptyset \)
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
Die roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def** Complement of event $E$, $E^C$

The event containing all outcomes in that are **not** in $E$.

$E^C = \{3, 4, 5, 6\}$
3 Axioms of Probability

Definition of probability: \( P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \)

Axiom 1: \( 0 \leq P(E) \leq 1 \)

Axiom 2: \( P(S) = 1 \)

Axiom 3: If \( E \) and \( F \) are mutually exclusive (\( E \cap F = \emptyset \)), then \( P(E \cup F) = P(E) + P(F) \)
Axiom 3 is the (analytically) useful Axiom

**Axiom 3:**

If $E$ and $F$ are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i)$$

(like the Sum Rule of Counting, but for probabilities)
Equally Likely Outcomes
Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

- Coin flip: \( S = \{ \text{Head, Tails} \} \)
- Flipping two coins: \( S = \{(H, H), (H, T), (T, H), (T, T)\} \)
- Roll of 6-sided die: \( S = \{1, 2, 3, 4, 5, 6\} \)

If we have equally likely outcomes, then

\[
P(\text{Each outcome}) = \frac{1}{|S|}
\]

Therefore

\[
P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{|E|}{|S|} \quad \text{(by Axiom 3)}
\]
Roll two dice

Roll two 6-sided fair dice. What is P(sum = 7)?

\[ S = \{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\
    (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\
    (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\
    (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\
    (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\
    (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \} \]

\[ E = \]

\[ P(E) = \frac{|E|}{|S|} \text{ Equally likely outcomes} \]
Target revisited
Target revisited

Let $E$ = the set of outcomes where you hit the target.

Screen size = 800 × 800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$
Target revisited

Let $E$ = the set of outcomes where you hit the target.

Screen size = $800 \times 800$

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

$$|S| = 800^2 \quad |E| \approx \pi \cdot 200^2$$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$
Not equally likely outcomes

Play the lottery.
What is \( P(\text{win}) \)?

\[
S = \{ \text{Lose, Win} \} \\
E = \{ \text{Win} \} \\
P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%
\]
Cats and sharks

4 cats and 3 sharks in a bag. 3 drawn. What is $P(1$ cat and $2$ sharks drawn)?

Note: Do indistinct objects give you an equally likely sample space?

(No)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Make indistinct items distinct to get equally likely outcomes.

A. $\frac{3}{7}$
B. $\frac{1}{4} \cdot \frac{2}{3}$
C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$
D. $\frac{12}{35}$
E. Zero/other
Cats and sharks (ordered solution)

4 cats and 3 sharks in a bag. 3 drawn.
What is $P(1$ cat and $2$ sharks drawn)?

Define

- $S = \text{Pick 3 distinct items}$
- $E = 1$ distinct cat, $2$ distinct sharks
Cats and sharks (unordered solution)

4 cats and 3 sharks in a bag. 3 drawn. What is P(1 cat and 2 sharks drawn)?

Define

- $S = \text{Pick 3 distinct items}$
- $E = \text{1 distinct cat, 2 distinct sharks}$

Make indistinct items distinct to get equally likely outcomes.

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes
03: Intro to Probability

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September 18, 2020
Holy cow, are all the pre-lecture videos going to be this long??

This course is **front-loaded with material/definitions**.

As we move into the latter half of the course, we will achieve a better balance.

Lecture Notes are a perfectly good substitute for videos if you learn better through reading.
The Count

Chance The Rapper
Summary so far

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1! n_2! \cdots n_r!}$

- Choose $k$ objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - $r$ groups: $\left(\begin{array}{c} n \\ n_1, n_2, \ldots, n_r \end{array}\right)$

- Put objects in $r$ buckets
  - Distinct
    - $r^n$
  - Indistinct
    - $\frac{(n + r - 1)!}{n! (r - 1)!}$

Equally likely outcomes:

$$P(E) = \frac{|E|}{|S|}$$

Combinatorics

Probability
Counting? Probability? Distinctness?

We choose 3 books from a set of 4 distinct (distinguishable) and 2 indistinct (indistinguishable) books. Each set of 3 books is equally likely.

Let event $E =$ our choice does not include both indistinct books.

1. How many distinct outcomes are in $E$?

2. What is $P(E)$?
Breakout Rooms for working through lecture exercises

- We may incorporate some of these during lecture
- You are always welcome to exit breakout rooms if you are more comfortable staying in the main room
- Turn on your camera if you are comfortable doing so
Think, then Breakout Rooms

Then check out the questions on the next slide (Slide 37). Post any clarifications here!

https://us.edstem.org/courses/2678/discussion/124598

Read both questions: 2 min

Breakout rooms: 5 min. Introduce yourself!
Poker Straights and Computer Chips

1. Consider equally likely 5-card poker hands.
   • Define “poker straight” as 5 consecutive rank cards of any suit
   What is P(Poker straight)?

2. Computer chips: $n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing.
   What is P(defective chip is in $k$ selected chips?)

Q1: odd-numbered breakout rooms
Q2: even-numbered breakout rooms
(if time, switch to other question)
1. Any Poker Straight

Consider equally likely 5-card poker hands.
• “straight” is 5 consecutive rank cards of any suit

What is \( P(\text{Poker straight}) \)?

Define
• \( S \) (unordered)
• \( E \) (unordered, consistent with \( S \))

Compute \( P(\text{Poker straight}) = \)
2. Chip defect detection

$n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing.

What is $P$ (defective chip is in $k$ selected chips?)

Define

- $S$ (unordered)
- $E$ (unordered, consistent with $S$)

Compute

$$P(E) =$$
2. Chip defect detection, solution #2

\( n \) chips are manufactured, 1 of which is defective. \( k \) chips are randomly selected from \( n \) for testing.

What is \( P(\text{defective chip is in } k \text{ selected chips?}) \)

Redefine experiment

1. Choose \( k \) indistinct chips (1 way)
2. Throw a dart and make one defective

Define

- \( S \) (unordered)
- \( E \) (unordered, consistent with \( S \))
Interlude for jokes/announcements
Announcements

Python tutorial
When: earlier today after class
Recorded/posted online

Geometric series, Integration by parts...

Resources on CS109 website
- Calculation Ref
- Python for Probability
- LaTeX Guides
- LaTeX Cheat Sheet
- Full Probability Reference (Overleaf)

Section sign-ups/
Acquaintance form
form link
Form due: Sat. 5:00pm 9/19
Results: latest Sunday

Office hours
Have started!
“The study finds that very few chords govern most of the music, a phenomenon that is also known in linguistics, where very few words dominate language corpora.... It characterizes Beethoven's specific composition style for the String Quartets, through a distribution of all the chords he used, how often they occur, and how they commonly transition from one to the other.”

Corollaries of Probability
3 Corollaries of Axioms of Probability

Corollary 1: \( P(E^C) = 1 - P(E) \)

Corollary 2: If \( E \subseteq F \), then \( P(E) \leq P(F) \)

Corollary 3: \( P(E \cup F) = P(E) + P(F) - P(EF) \)
(Inclusion-Exclusion Principle for Probability)
Selecting Programmers

- $P(\text{student programs in Python}) = 0.28$
- $P(\text{student programs in C++}) = 0.07$
- $P(\text{student programs in Python and C++}) = 0.05$.

What is $P(\text{student does not program in (Python or C++)})$?

1. Define events & state goal
2. Identify known probabilities
3. Solve
Inclusion-Exclusion Principle (Corollary 3)

**Corollary 3:**

\[ P(E \cup F) = P(E) + P(F) - P(EF) \]

**General form:**

\[ P\left( \bigcup_{i=1}^{n} E_i \right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \ldots < i_r} P\left( \bigcap_{j=1}^{r} E_{i_j} \right) \]

(see Lecture Notes)

\[ P(E \cup F \cup G) = \]

- \( r = 1: \quad P(E) + P(F) + P(G) \)
- \( r = 2: \quad -P(E \cap F) - P(E \cap G) - P(F \cap G) \)
- \( r = 3: \quad +P(E \cap F \cap G) \)
Takeaway: Union of events

Axiom 3, Mutually exclusive events

Corollary 3, Inclusion-Exclusion Principle

The challenge of probability is in defining events. Some event probabilities are easier to compute than others.
Serendipity

Let it find you.

SERENDIPITY
the effect by which one accidentally stumbles upon something truly wonderful, especially while looking for something entirely unrelated.

WHEN YOU MEET YOUR BEST FRIEND
Somewhere you didn't expect to.
Serendipity

- The population of Stanford is \( n = 17,000 \) people.
- You are friends with \( r \) people.
- Walk into a room, see \( k = 200 \) random people.
- Assume each group of \( k \) Stanford people is equally likely to be in room.

What is the probability that you see someone you know in the room?

http://web.stanford.edu/class/cs109/demos/serendipity.html
Slide 52 is a question to think over by yourself (~2min).

Post any clarifications here or in chat!

https://us.edstem.org/courses/2678/discussion/124109
Serendipity

The population of Stanford is \( n = 17,000 \) people.

You are friends with \( r = 100 \) people.

Walk into a room, see \( k = 223 \) random people.

Assume each group of \( k \) Stanford people is equally likely to be in room.

What is the probability that you see at least one friend in the room?

Define

- \( S \) (unordered)
- \( E: \geq 1 \) friend in the room

What strategy would you use?

A. \( P(\text{exactly 1}) + P(\text{exactly 2}) + P(\text{exactly 3}) + \cdots \)

B. \( 1 - P(\text{see no friends}) \)
Serendipity

• The population of Stanford is $n = 17,000$ people.
• You are friends with $r = 100$ people.
• Walk into a room, see $k = 223$ random people.
• Assume each group of $k$ Stanford people is equally likely to be in room.

What is the probability that you see at least one friend in the room?

Define

• $S$ (unordered)
• $E: \geq 1$ friend in the room

It is often much easier to compute $P(E^c)$. 
The Birthday Paradox Problem

What is the probability that in a set of $n$ people, at least one pair of them will share the same birthday?

For you to think about (and discuss in section!)
Card Flipping

In a 52-card deck, cards are flipped one at a time. After the first ace (of any suit) appears, consider the next card. Is $P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = \text{2 Clubs})$?

Once you think you have an answer, vote on our Zoom poll:

https://us.edstem.org/courses/2678/discussion/124598

Check out the Lecture Notes!
Have a good weekend!