06: Random Variables

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Probability of events

E or F
\[ P(E \cup F) \]

E and F
\[ P(EF) \]

Just add!
\[ P(E) + P(F') \]

Mutually exclusive?

Inclusion-Exclusion Principle
\[ P(E) + P(F) - P(E \cap F) \]

Independent?

Just multiply!
\[ P(E)P(F) \]

De Morgan’s

Just add!
\[ P(E) + P(F) \]

Chain Rule
\[ P(E)P(F|E) \]
\[ P(F)P(E|F) \]
Hash table fun

- $m$ strings are hashed (unequally) into a hash table with $n$ buckets.
- Each string hashed is an independent trial w.p. $p_i$ of getting hashed into bucket $i$.

1. $E$ = bucket 1 has $\geq 1$ string hashed into it.

What is $P(E)$?

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$
$$= 1 - P(((S_1 \cup S_2 \cup \cdots \cup S_m)^C)$$
$$= 1 - P(S_1^C S_2^C \cdots S_m^C)$$
$$= 1 - P(S_1^C)P(S_2^C)\cdots P(S_m^C) = 1 - (P(S_1^C))^m$$
$$= 1 - (1 - p_1)^m$$

Define

- $S_i$ = string $i$ is hashed into bucket 1
- $S_i^C$ = string $i$ is not hashed into bucket 1

$P(S_i) = p_1$
$P(S_i^C) = 1 - p_1$

$S_i$ independent trials
The fun never stops with hash tables

- $m$ strings are hashed (unequally) into a hash table with $n$ buckets.
- Each string hashed is an independent trial w.p. $p_i$ of getting hashed into bucket $i$.

1. $E = \text{bucket 1 has } \geq 1 \text{ string hashed into it.}$
2. $E = \text{at least 1 of buckets 1 to } k \text{ has } \geq 1 \text{ string hashed into it.}$
3. $E = \text{each of buckets 1 to } k \text{ has } \geq 1 \text{ string hashed into it.}$

What is $P(E)$?

Define $F_i = \text{bucket } i \text{ has at least one string in it}$
The fun never stops with hash tables

- \( m \) strings are hashed (unequally) into a hash table with \( n \) buckets.
- Each string hashed is an independent trial w.p. \( p_i \) of getting hashed into bucket \( i \).

1. \( E = \) bucket 1 has \( \geq 1 \) string hashed into it.
2. \( E = \) at least 1 of buckets 1 to \( k \) has \( \geq 1 \) string hashed into it.
3. \( E = \) each of buckets 1 to \( k \) has \( \geq 1 \) string hashed into it.

What is \( P(E) \)?

\[
P(E) = P(F_1F_2 \cdots F_k) \\
= 1 - P((F_1F_2 \cdots F_k)^c) \\
= 1 - P(F_1^c \cup F_2^c \cup \cdots \cup F_k^c) \\
= 1 - \sum_{i=1}^{k} (-1)^{r+1} \sum_{i_1<\cdots<i_r} P(F_{i_1}^cF_{i_2}^c \cdots F_{i_r}^c)
\]

where \( P(F_{i_1}^cF_{i_2}^c \cdots F_{i_r}^c) = (1 - p_{i_1} - p_{i_2} \cdots - p_{i_r})^m \)

Define \( F_i = \) bucket \( i \) has at least one string in it.
Today’s plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation
Conditional Paradigm

For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

Axiom 1

\[ 0 \leq P(A|E) \leq 1 \]

Corollary 1 (complement)

\[ P(A|E) = 1 - P(A^C|E) \]

Commutativity

\[ P(AB|E) = P(BA|E) \]

Chain Rule

\[ P(AB|E) = P(B|E)P(A|BE) \]

Bayes’ Theorem

\[ P(A|BE) = \frac{P(B|AE)P(A|E)}{P(B|E)} \]
Conditional Independence

Two events $A$ and $B$ are defined as conditionally independent given $E$ if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$
Netflix and Condition

Let $E$ = a user watches Life is Beautiful.
Let $F$ = a user watches Amelie.
What is $P(E)$?

$$P(E) \approx \frac{\# \text{ people who have watched movie}}{\# \text{ people on Netflix}} = \frac{10,234,231}{50,923,123} \approx 0.20$$

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\# \text{ people who have watched both}}{\# \text{ people who have watched Amelie}} \approx 0.42$$
Netflix and Condition

Let $E$ be the event that a user watches the given movie.
Let $F$ be the event that the same user watches Amelie.

$P(E) = 0.19$ \hspace{2cm} $P(E) = 0.32$

$P(E|F) = 0.14$ \hspace{2cm} $P(E|F) = 0.35$

$P(E) = 0.20$ \hspace{2cm} $P(E) = 0.09$ \hspace{2cm} $P(E) = 0.20$

$P(E|F) = 0.20$ \hspace{2cm} $P(E|F) = 0.72$ \hspace{2cm} $P(E|F) = 0.42$

Independent!
What if $E_1E_2E_3E_4$ are not independent? (e.g., all international emotional comedies)

$$P(E_4|E_1E_2E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)}$$
Netflix and Condition

What if $E_1 E_2 E_3 E_4$ are conditionally independent $K$?

- $K$: likes international emotional comedies
- $E_1$: watched "Amélie"
- $E_2$: watched "Life is Beautiful"
- $E_3$: watched "3 Idiots"
- $E_4$: watched "Nairobi Half Life"

Will they watch?

$$P(E_4|E_1 E_2 E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)}$$

$$P(E_4|E_1 E_2 E_3 K) = P(E_4|K)$$
Netflix and Condition

\[ K: \text{likes international emotional comedies} \]

\[ E_1, E_2, E_3, E_4 \] are dependent

\[ E_1 E_2 E_3 E_4 \] are conditionally independent given \( K \)

Dependent events can become conditionally independent.
Roll two 6-sided dice, yielding values $D_1$ and $D_2$.

Let event $E$: $D_1 = 1$

event $F$: $D_2 = 6$

event $G$: $D_1 + D_2 = 7$

$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are $E$ and $F$ independent?

\[
P(E) = 1/6 \quad P(F) = 1/6 \quad P(EF) = 1/36
\]

2. Are $E$ and $F$ independent given $G$?

Independent events can become conditionally dependent.
The beauty of conditional independence

Generalized Chain Rule:

\[ P(E_1E_2E_3 \ldots E_nF) = \]

\[ P(F)P(E_1|F)P(E_2|E_1F)P(E_3|E_1E_2F) \ldots P(E_n|E_1E_2 \ldots E_{n-1}F) \]

If \( E_1, E_2, \ldots, E_n \) are all conditionally independent given \( F \):

\[ P(E_1E_2E_3 \ldots E_nF) = P(F)P(E_1|F)P(E_2|F) \ldots P(E_n|F) \]

More on this in a future lecture!
Conditional independence is a Big Deal

Conditional independence is a practical, real-world way of decomposing hard probability questions.

“Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory.”

– Judea Pearl wins 2011 Turing Award, “For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning”

Independence relationships can change with conditioning.

- A and B independent does NOT necessarily mean A and B independent given E.
Today’s plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation
A random variable is a real-valued function defined on a sample space.

Example:

3 coins are flipped. Let $X = \# \text{ of heads}$. $X$ is a random variable.

1. What is the value of $X$ for the outcomes:
   • $(T,T,T)$?
   • $(H,H,T)$?

2. What is the event (set of outcomes) where $X = 2$?

3. What is $P(X = 2)$?
Random variables are **NOT** events!

It is confusing that random variables and events use the same notation.

- **Random variables ≠ events.**
- We can define an event to be a particular assignment of a random variable.

<table>
<thead>
<tr>
<th>$X = x$</th>
<th>$P(X = x)$</th>
<th>Set of outcomes</th>
<th>Possible event $E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>1/8</td>
<td>{(T, T, T)}</td>
<td>Flip 0 heads</td>
</tr>
<tr>
<td>$X = 1$</td>
<td>3/8</td>
<td>{(H, T, T), (T, H, T), (T, T, H)}</td>
<td>Flip exactly 1 head</td>
</tr>
<tr>
<td>$X = 2$</td>
<td>3/8</td>
<td>{(H, H, T), (H, T, H), (T, H, H)}</td>
<td>The event where $X = 2$</td>
</tr>
<tr>
<td>$X = 3$</td>
<td>1/8</td>
<td>{(H, H, H)}</td>
<td>Flip 0 tails</td>
</tr>
<tr>
<td>$X \geq 4$</td>
<td>0</td>
<td>{}</td>
<td>Flip 4 or more heads</td>
</tr>
</tbody>
</table>
Example random variable

Consider 5 flips of a coin which comes up heads with probability $p$.

- Each coin flip is an independent trial.
- Recall $P(2 \text{ heads}) = \binom{5}{2} p^2 (1 - p)^3$, $P(3 \text{ heads}) = \binom{5}{3} p^3 (1 - p)^2$

Let $Y = \# \text{ of heads on 5 flips}$.

1. What is the range of $Y$?
   In other words, what are the values that $Y$ can take on with non-zero probability?

2. What is $P(Y = k)$, where $k$ is in the range of $Y$?
Today’s plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation
Probability Mass Function (PMF)

\[ Y = 2 \]

- **Event**: \( P(Y = 2) \)
- **Probability**: (number b/t 0 and 1)

\[ P(Y = k) \]

- **Function** on \( k \) with range 0 and 1

**Random variable** (e.g., # of heads in 5 coin flips, unbiased coin)
Discrete RVs and Probability Mass Functions

A random variable $X$ is **discrete** if its range has countably many values.
- $X = x$, where $x \in \{x_1, x_2, x_3, \ldots\}$

The **probability mass function** (PMF) of a discrete random variable is

$$P(X = x) = p(x) = p_X(x)$$

**shorthand notation**

- Probabilities must sum to 1:

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

This last bullet is a good way to verify any PMF you create.
PMF for a single 6-sided die

Let $X$ be a random variable that represents the result of a single dice roll.

- Range of $X$ : $\{1, 2, 3, 4, 5, 6\}$
- Therefore $X$ is a discrete random variable.
- PMF of $X$:
  \[
p(x) = \begin{cases} 
  \frac{1}{6} & x \in \{1, \ldots, 6\} \\
  0 & \text{otherwise}
\end{cases}
\]
PMF for the sum of two dice

Let $Y$ be a random variable that represents the sum of two independent dice rolls.

Range of $Y$: \{2, 3, ..., 11, 12\}

$$p(y) = \begin{cases} \frac{y-1}{36} & y \in \mathbb{Z}, 2 \leq y \leq 6 \\ \frac{13-y}{36} & y \in \mathbb{Z}, 7 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

Check:

$$\sum_{y=2}^{12} p(y) = 1$$

$P(Y = y)$
Problem Set 1
Due: an hour ago
On-time grades: next Friday
Solutions: next Friday

Problem Set 2
Out: today
Due: Monday 1/27
Covers: through today
For a random variable $X$, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{ where } -\infty < a < \infty$$

For a discrete RV $X$, the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$
Let $X$ be a random variable that represents the result of a single dice roll.

The probability mass function (PMF) of $X$ is given by:

- $P(X = 0) = \frac{1}{6}$
- $P(X = 1) = \frac{1}{6}$
- $P(X = 2) = \frac{1}{6}$
- $P(X = 3) = \frac{1}{6}$
- $P(X = 4) = \frac{1}{6}$
- $P(X = 5) = \frac{1}{6}$
- $P(X = 6) = \frac{1}{6}$

The cumulative distribution function (CDF) of $X$ is given by:

- $F(0) = P(X \leq 0) = 0$
- $F(1) = P(X \leq 1) = \frac{1}{6}$
- $F(2) = P(X \leq 2) = \frac{2}{6}$
- $F(3) = P(X \leq 3) = \frac{3}{6}$
- $F(4) = P(X \leq 4) = \frac{4}{6}$
- $F(5) = P(X \leq 5) = \frac{5}{6}$
- $F(6) = P(X \leq 6) = 1$

The CDF is a step function that increases by $\frac{1}{6}$ at each integer value of $X$.
Today’s plan

Conditional Independence

Random Variables

PMFs and CDFs

Expectation
Expectation

The expectation of a discrete random variable $X$ is defined as:

$$E[X] = \sum_{x : p(x) > 0} p(x) \cdot x$$

• Note: sum over all values of $X = x$ that have non-zero probability.

• Other names: mean, expected value, weighted average, center of mass, first moment
What is the expected value of a 6-sided die roll?

1. Define random variables

\[ X = \text{RV for value of roll} \]

\[ P(X = x) = \begin{cases} 1/6 & x \in \{1, \ldots, 6\} \\ 0 & \text{otherwise} \end{cases} \]

2. Solve

\[ E[X] = 1 \left( \frac{1}{6} \right) + 2 \left( \frac{1}{6} \right) + 3 \left( \frac{1}{6} \right) + 4 \left( \frac{1}{6} \right) + 5 \left( \frac{1}{6} \right) + 6 \left( \frac{1}{6} \right) = \frac{7}{2} \]
Lying with statistics

“There are three kinds of lies: lies, damned lies, and statistics”
–popularized by Mark Twain, 1906
Lying with statistics

A school has 3 classes with 5, 10, and 150 students. What is the average class size?
A school has 3 classes with 5, 10, and 150 students. What is the average class size?

1. Interpretation #1
   - Randomly choose a class with equal probability.
   - $X =$ size of chosen class
   \[
   E[X] = 5 \left( \frac{1}{3} \right) + 10 \left( \frac{1}{3} \right) + 150 \left( \frac{1}{3} \right) \\
   = \frac{165}{3} = 55
   \]

2. Interpretation #2
   - Randomly choose a student with equal probability.
   - $Y =$ size of chosen class
   \[
   E[Y] = 5 \left( \frac{5}{165} \right) + 10 \left( \frac{10}{165} \right) + 150 \left( \frac{150}{165} \right) \\
   = \frac{22635}{165} \approx 137
   \]
Important properties of expectation

1. Linearity:
   \[ E[aX + b] = aE[X] + b \]

2. Expectation of a sum = sum of expectation:
   \[ E[X + Y] = E[X] + E[Y] \]

3. Law of the unconscious statistician (LOTUS):
   \[ E[g(X)] = \sum_x g(x)p(x) \]

Let \( X \) = 6-sided dice roll, \( Y = 2X - 1 \).
- \( E[X] = 3.5 \)
- \( E[Y] = 6 \)

Sum of two dice rolls:
- Let \( X = \) roll of die 1
  \( Y = \) roll of die 2
- \( E[X + Y] = 3.5 + 3.5 = 7 \)
Being a statistician unconsciously

Let $X$ be a discrete random variable.

\begin{itemize}
  \item $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$
\end{itemize}

Let $Y = |X|$. What is $E[Y]$?

\[
E[g(X)] = \sum_x g(x)p(x)
\]

Expectation of $g(X)$