CS109: Probability for Computer Scientists

Lecture 10 — Probabilistic Models

January 28

Relative Probability of Continuous Variables

X is the time it takes for a student to complete a problem set. $X \sim N(\mu = 10, \sigma^2 = 2)$. How much more likely is the student to complete the problem set in 10 hours than 5 hours?

We compare the relative likelihoods using the probability density function multiplied by epsilon:

$$\frac{P(X=10)}{P(X=5)} = \frac{\epsilon_X \cdot f_X(10)}{\epsilon_X \cdot f_X(5)}$$

Then we can cancel the ϵ_X, leaving us with just the ratio of probability densities.

$$\frac{P(X=10)}{P(X=5)} = \frac{f_X(10)}{f_X(5)}$$

For a normal distribution,

$$f_X(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Thus,

$$\frac{f_X(10)}{f_X(5)} = \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{(10-10)^2}{4}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(5-10)^2}{4}}} = \frac{e^0}{e^{-25/4}} = 518$$

So completing the problem set in 10 hours is 518 times more likely than completing it in 5 hours.

Dating Status

Problem 2: Dating Status

Let X be a student's dating status and Y be their year in school. The joint distribution is given below.

Year	**Single**	**Relationship**	**Complicated**
Frosh	0.13	0.08	0.02
Soph	0.17	0.11	0.02
Junior	0.09	0.10	0.02
Senior	0.02	0.07	0.01
5+	0.06	0.09	0.04

(a) What is the probability that a randomly selected student is in a relationship?

(b) What is the probability that a randomly selected student is a Frosh?

(a) We use the law of total probability:

$$P(X = \text{Relationship}) = \sum_y P(X = \text{Relationship}, Y = y)$$

Summing the Relationship column:

$$P(X = \text{Relationship}) = 0.08 + 0.11 + 0.10 + 0.07 + 0.09 = 0.45$$

(b) We again use the law of total probability:

$$P(Y = \text{Frosh}) = \sum_x P(X = x, Y = \text{Frosh})$$

Summing the Frosh row:

$$P(Y = \text{Frosh}) = 0.13 + 0.08 + 0.02 = 0.23$$

I Heard That — Discrete

Let X be the change in gaze (measured in degrees) over 3 seconds after a sound is played. Let Y be whether the baby can hear the sound:

$$Y = \begin{cases} 1 & \text{baby can hear the sound} \\ 0 & \text{baby cannot hear the sound} \end{cases}$$

Suppose $P(Y = 1) = \frac{3}{4}$ You also have the following information:

Value of X	$P(X \mid Y = 1)$	$P(X \mid Y = 0)$
0 to 5	0.08	0.40
5 to 10	0.15	0.30
10 to 15	0.35	0.12
15 to 20	0.20	0.08
20 to 25	0.12	0.05
Above 25	0.10	0.05

You observe $X = 0$ (i.e., the gaze change falls in the 0 to 5 bin). What is the probability that the baby can hear the sound?

We apply Bayes' rule:

$$P(Y=1 \mid X=0) = \frac{P(X=0 \mid Y=1)P(Y=1)}{P(X=0 \mid Y=1)P(Y=1) + P(X=0 \mid Y=0)P(Y=0)}$$

Plug in values from the table and priors:

$$P(Y=1 \mid X=0) = \frac{0.08 \cdot 0.75}{0.08 \cdot 0.75 + 0.40 \cdot 0.25} = \frac{0.06}{0.06 + 0.10} = \frac{3}{8}$$

I Heard That — Continuous

Normal Assumption: Let X be the change in gaze (in degrees) after a sound is played. Let $Y = 1$ indicate the baby can hear the sound and $Y = 0$ otherwise. Assume the same prior $P(Y = 1) = \frac{3}{4}$.

$$X \mid Y = 1 \sim \mathcal{N}(\mu = 15, \sigma^2 = 25)$$

$$X \mid Y = 0 \sim \mathcal{N}(\mu = 8, \sigma^2 = 25)$$

You observe a new baby with $X = 14$. What is your belief that the baby can hear the sound?

We apply Bayes' rule using probability densities:

$$P(Y=1 \mid X=14) = \frac{f(X=14|Y=1)P(Y=1)}{f(X=14|Y=1)P(Y=1)+f(X=14|Y=0)P(Y=0)}$$

The normal pdf is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Here $\sigma^2 = 25$, so $\sigma = 5$.

$$f(X=14|Y=1) = \frac{1}{\sqrt{2\pi \cdot 25}} \exp\left(-\frac{(14-15)^2}{2 \cdot 25}\right)$$

$$f(X=14|Y=0) = \frac{1}{\sqrt{2\pi \cdot 25}} \exp\left(-\frac{(14-8)^2}{2 \cdot 25}\right)$$

Plugging into Bayes' rule (with $P(Y=1) = 0.75$, $P(Y=0) = 0.25$):

$$P(Y=1 \mid X=14) = \frac{\left(\frac{1}{\sqrt{2\pi\cdot 25}} \exp\left(-\frac{(14-15)^2}{50}\right)\right) \cdot 0.75}{\left(\frac{1}{\sqrt{2\pi\cdot 25}} \exp\left(-\frac{(14-15)^2}{50}\right)\right) \cdot 0.75 + \left(\frac{1}{\sqrt{2\pi\cdot 25}} \exp\left(-\frac{(14-8)^2}{50}\right)\right) \cdot 0.25}$$

The $\frac{1}{\sqrt{2\pi\cdot 25}}$ term appears in both numerator and denominator, so it cancels:

$$P(Y=1 \mid X=14) = \frac{0.75 \exp\left(-\frac{(14-15)^2}{50}\right)}{0.75 \exp\left(-\frac{(14-15)^2}{50}\right) + 0.25 \exp\left(-\frac{(14-8)^2}{50}\right)}$$

$$P(Y=1 \mid X=14) = \frac{0.75\, e^{-1/50}}{0.75\, e^{-1/50} + 0.25\, e^{-36/50}} \approx 0.86$$

So under the normal assumption, we believe there is about an 86% chance the baby can hear the sound.

Bayesian Carbon Dating

Let A be the age of a sample in years (e.g., $A = 100$ means 100 years old). Let M be the number of C14 molecules remaining in the sample.

Assume there were originally 1000 C14 molecules, and you observe $M = 900$.

Assume a uniform prior on age over integers $i \in \{100, 101, \ldots, 10000\}$:

$$P(A = i) = \frac{1}{9901}.$$

Each molecule decays independently. If the sample is age i, then each molecule remains with probability

$$p_i = P(T > i), \quad \text{where } T \sim \text{Exp}(\lambda = 1/8267).$$

(a) Write an expression for your updated belief in age taking on any value in range 100 to 1000 after observing 900 molecules of C14.

(b) Now write an expression for the likelihood of observing exactly 900 molecules given you know the age of the object.

(a) By Bayes' rule, for any $i \in \{100, \dots, 10000\}$,

$$P(A = i \mid M = 900) = \frac{P(M = 900 \mid A = i)\, P(A = i)}{P(M = 900)}.$$

$$P(M = 900) = \sum_{j=100}^{10000} P(M = 900 \mid A = j)\, P(A = j).$$

(b) Given age $A = i$, each of the 1000 molecules remains independently with probability p_i, so

$$M \mid A = i \sim \text{Bin}(n = 1000, p = p_i),$$

and

$$P(M = 900 \mid A = i) = \binom{1000}{900} (p_i)^{900} (1 - p_i)^{100}.$$

Each molecule's time to decay is exponential:

$$T \sim \text{Exp}(\lambda = 1/8267),$$

so the probability a molecule survives past time i is

$$p_i = P(T > i) = e^{-i/8267}.$$

Substituting into the binomial likelihood:

$$P(M = 900 \mid A = i) = \binom{1000}{900} \left(e^{-i/8267}\right)^{900} \left(1 - e^{-i/8267}\right)^{100}.$$