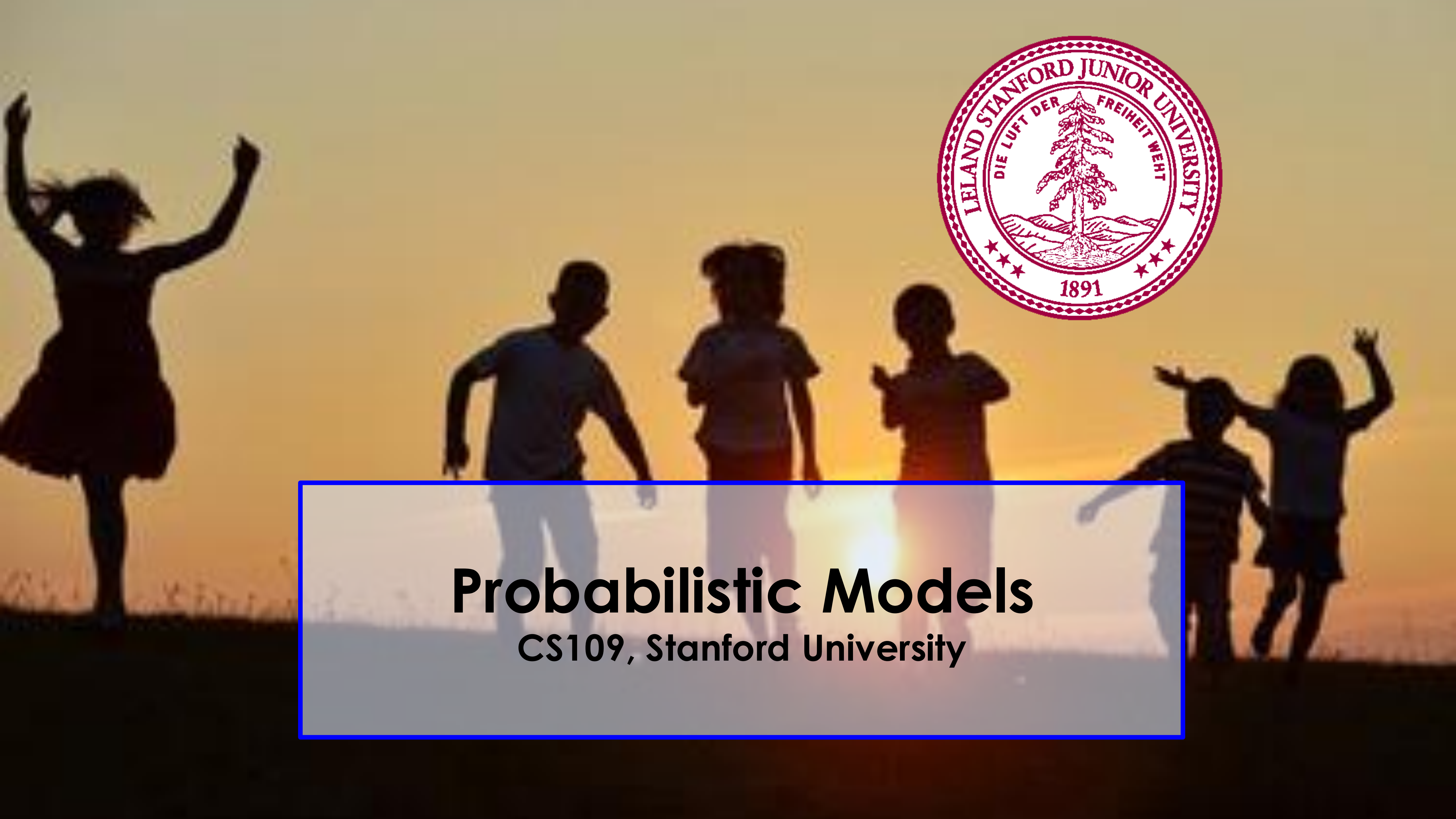


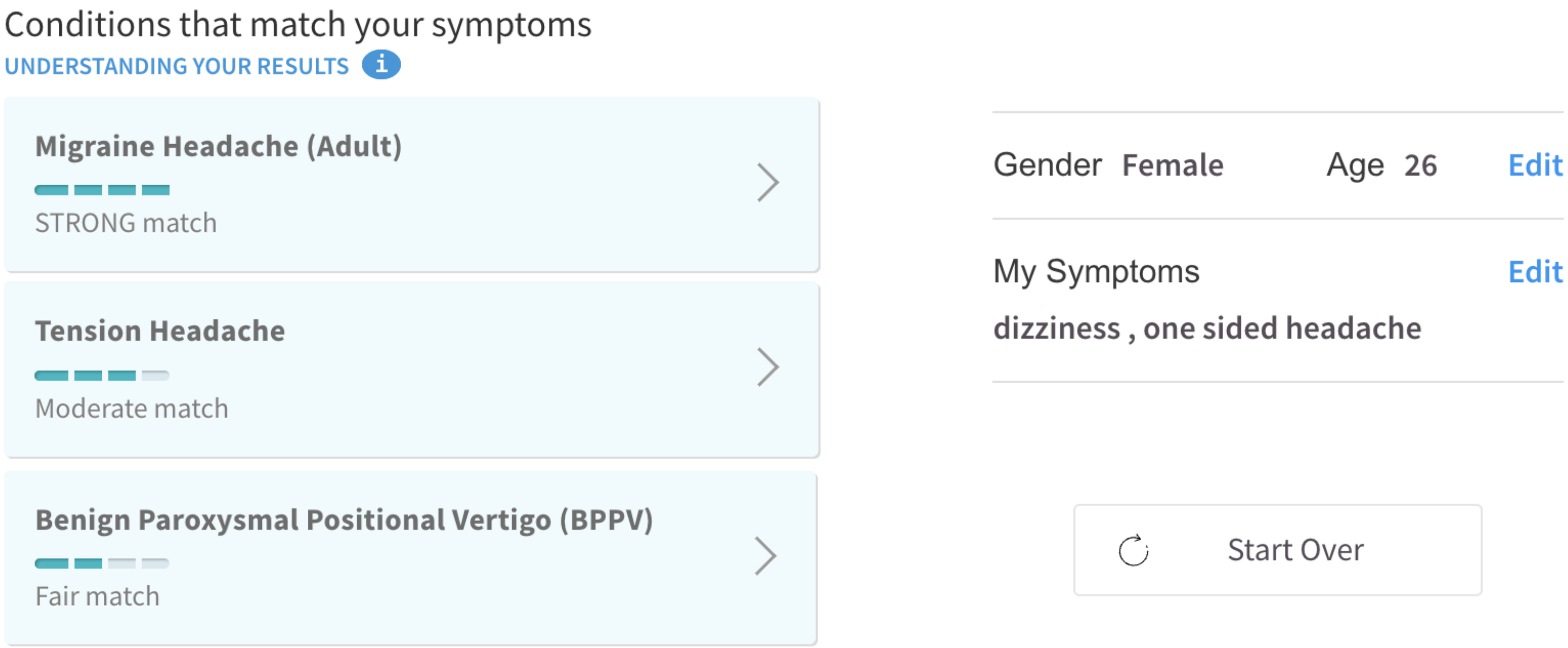
# Probabilistic Models

CS109, Stanford University

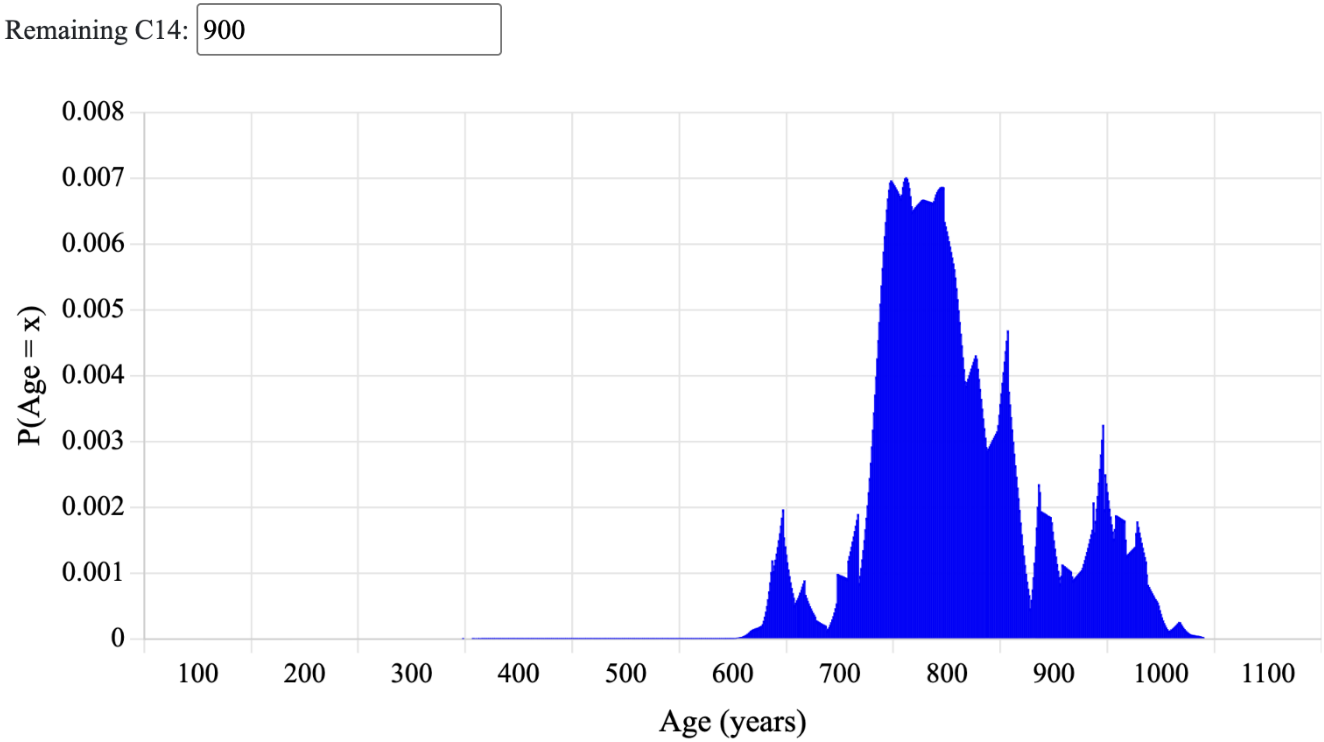
Sign up for PEP!!!

# Multiple Random Variables. Start of Digital Revolution

# Multiple Random Variables. Start of Digital Revolution

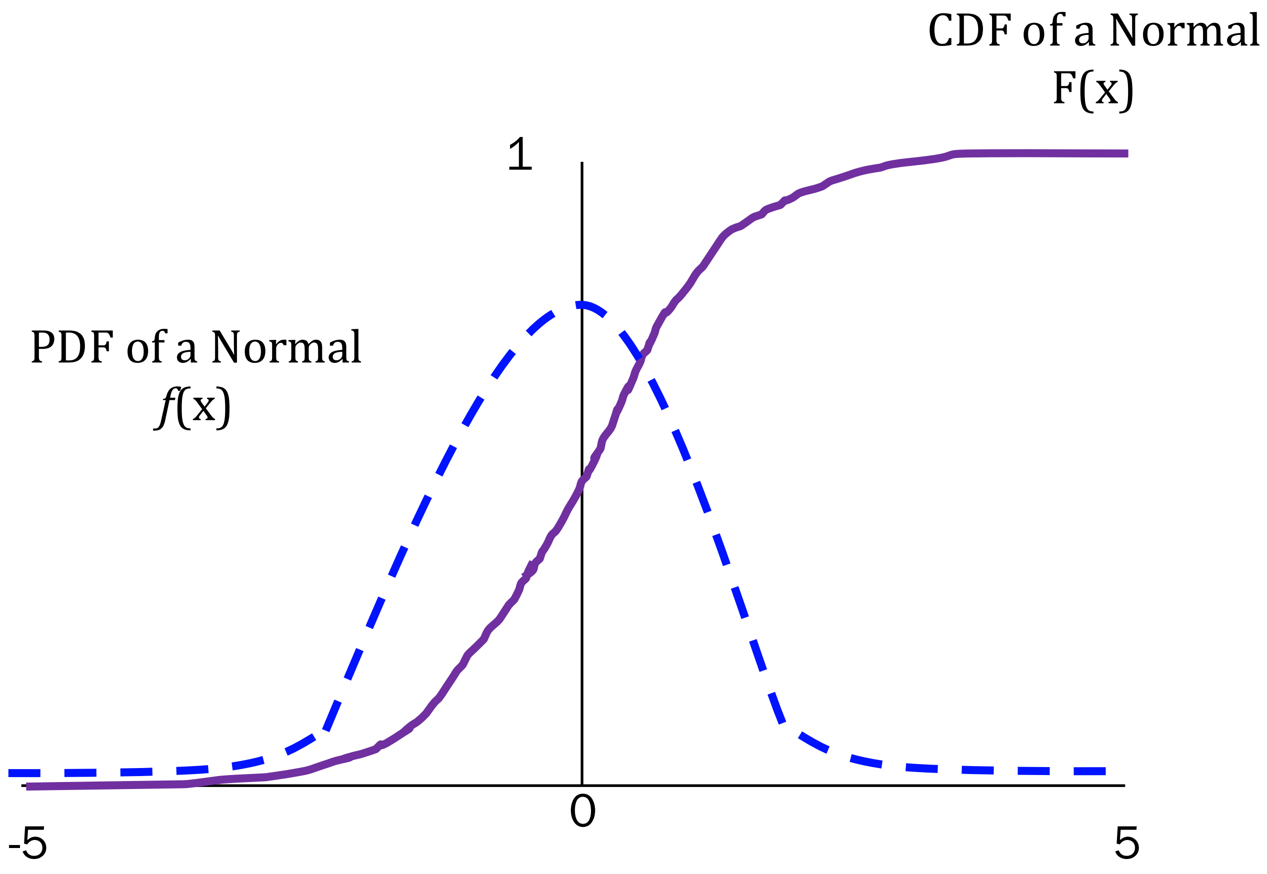


# Bayesian Carbon Dating



# First, some review

# Normal Distribution




$f(x)$ = derivative of probability

$F(x) = P(X < x)$

# Cumulative Distribution Function (CDF)

$\mathcal{N}(\mu, \sigma^2)$

CDF of Standard Normal: A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

Table of $\Phi(z)$ values are precomputed

# Probability Density Function (PDF)

$$\mathcal{N}(\mu, \sigma^2)$$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at *x*

a constant

sigma shows up twice

How does python sample from a Gaussian?

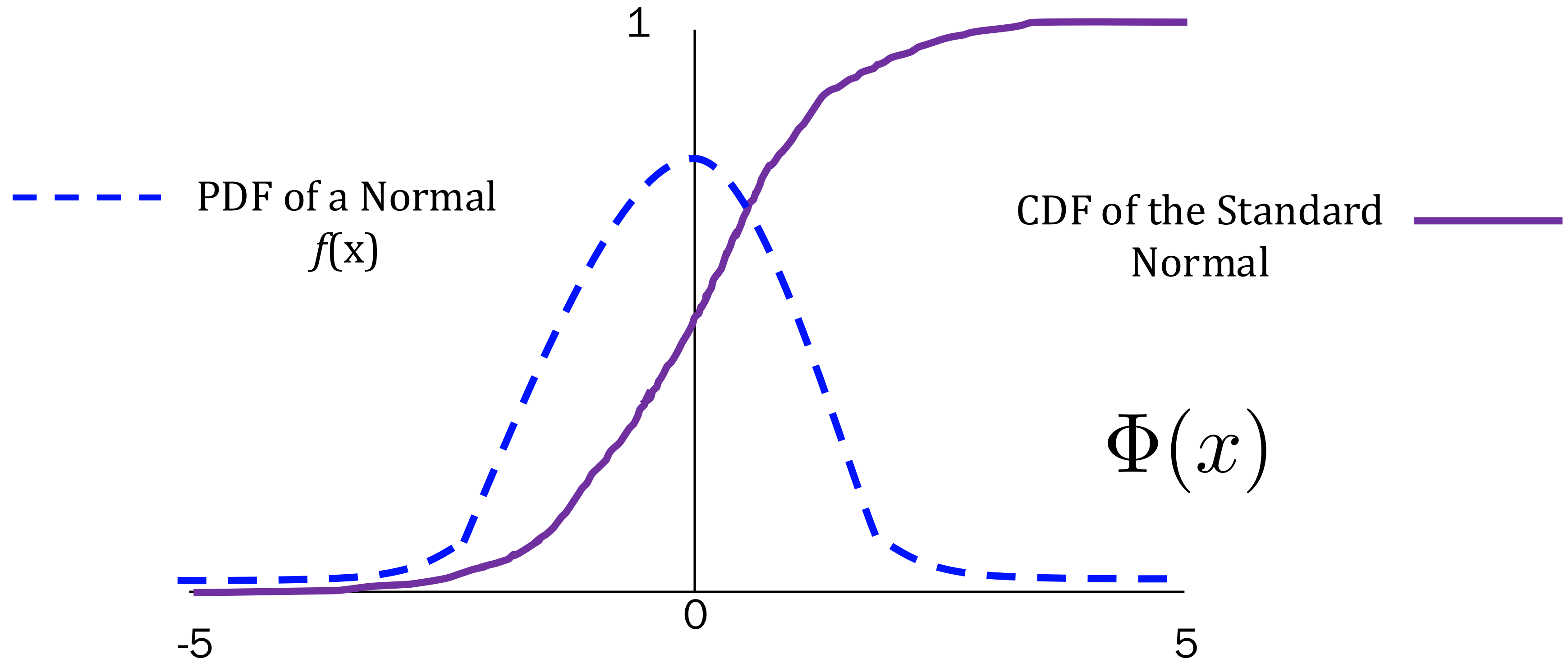
```
from random import *

for i in range(10):
  mean = 5
  std = 1
  sample = gauss(mean, std)
  print sample
```

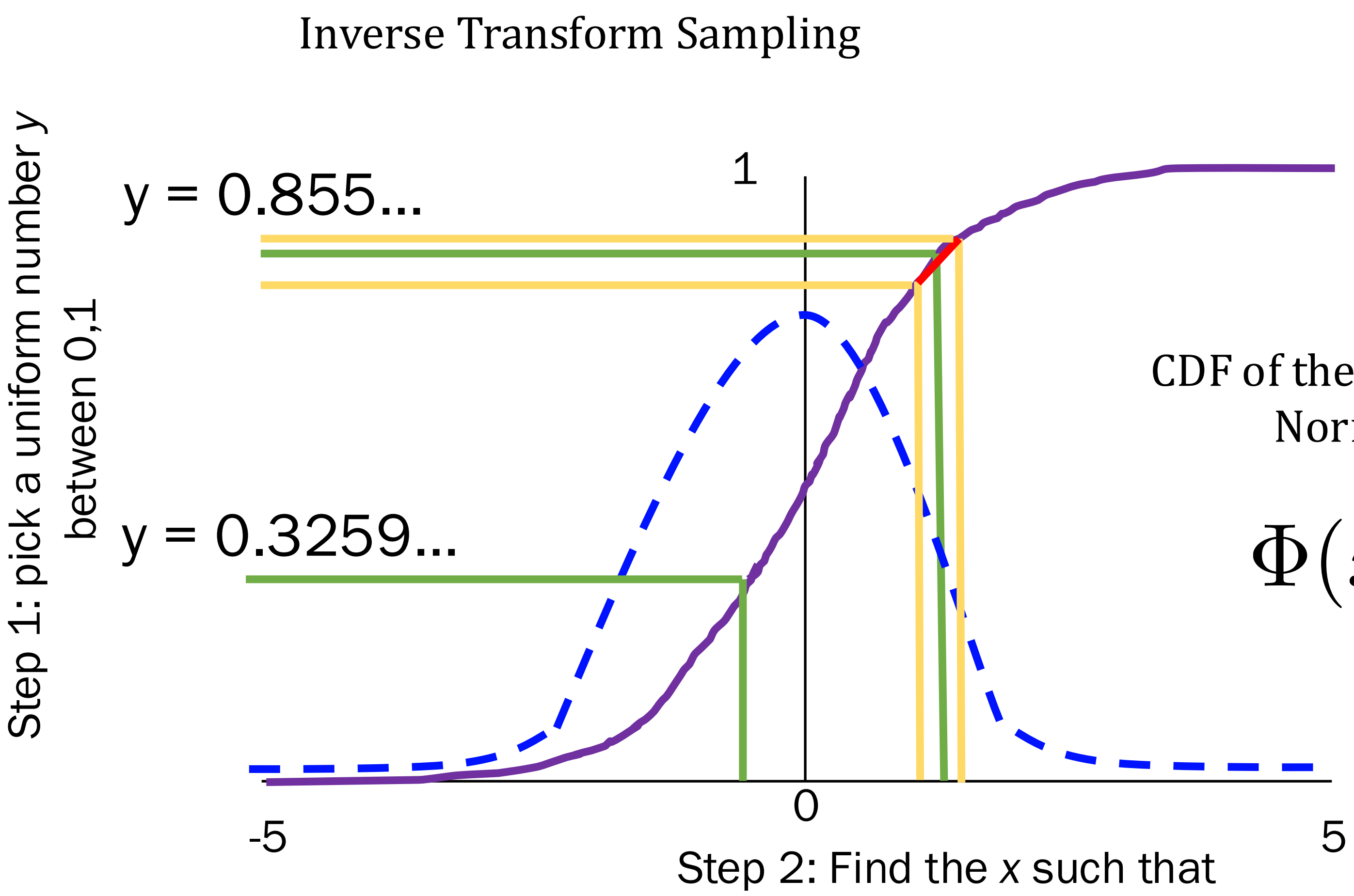
How does this work?

3.79317794179
5.19104589315
4.209360629
5.39633891584
7.10044176511
6.72655475942
5.51485158841
4.94570606131
6.14724644482
4.73774184354

# How Does a Computer Sample a Normal?



# How Does a Computer Sample a Normal?




$$\Phi(x) = y$$

$$x = \Phi^{-1}(y)$$

CDF of the Standard Normal

$$\Phi(x)$$

Sample 1:
1.201234

Sample 2:
-0.45123

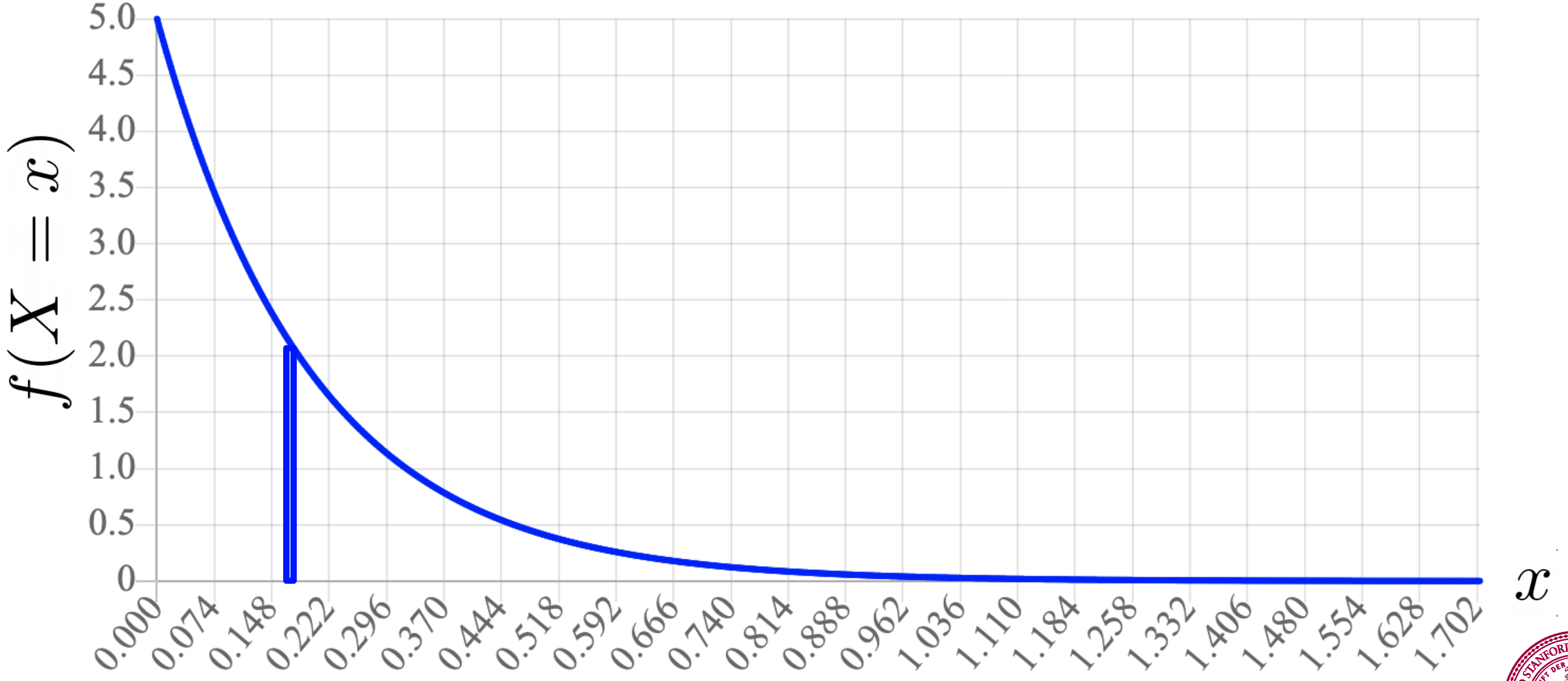
Further reading: Box–Muller transform

End of review

# A New Perspective

# Epsilon: Useful perspective

$$P(X = x) = f(X = x) \cdot \epsilon_x$$

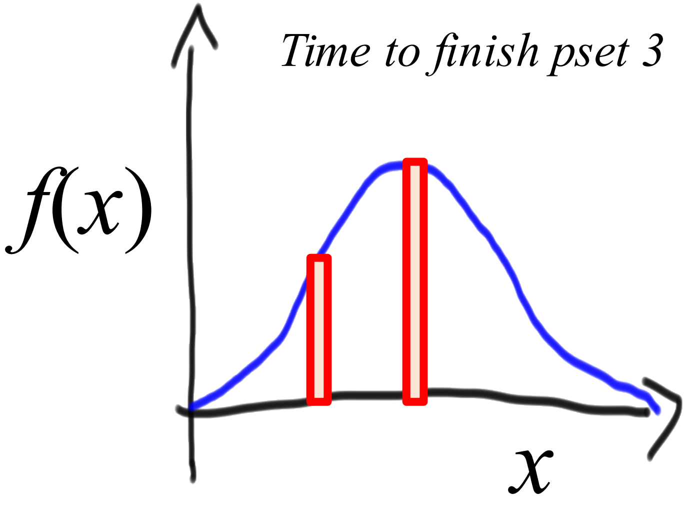



* In the limit as $\epsilon_x$ goes to zero

# Relative Probability of Continuous Variables

$X$ = time to finish pset 3

$X \sim N(\mu = 10, \sigma^2 = 2)$




How much more likely are you to complete pset 3 in 10 hours than in 5?

$$\frac{P(X=10)}{P(X=5)} = \frac{\varepsilon f(X=10)}{\varepsilon f(X=5)}$$

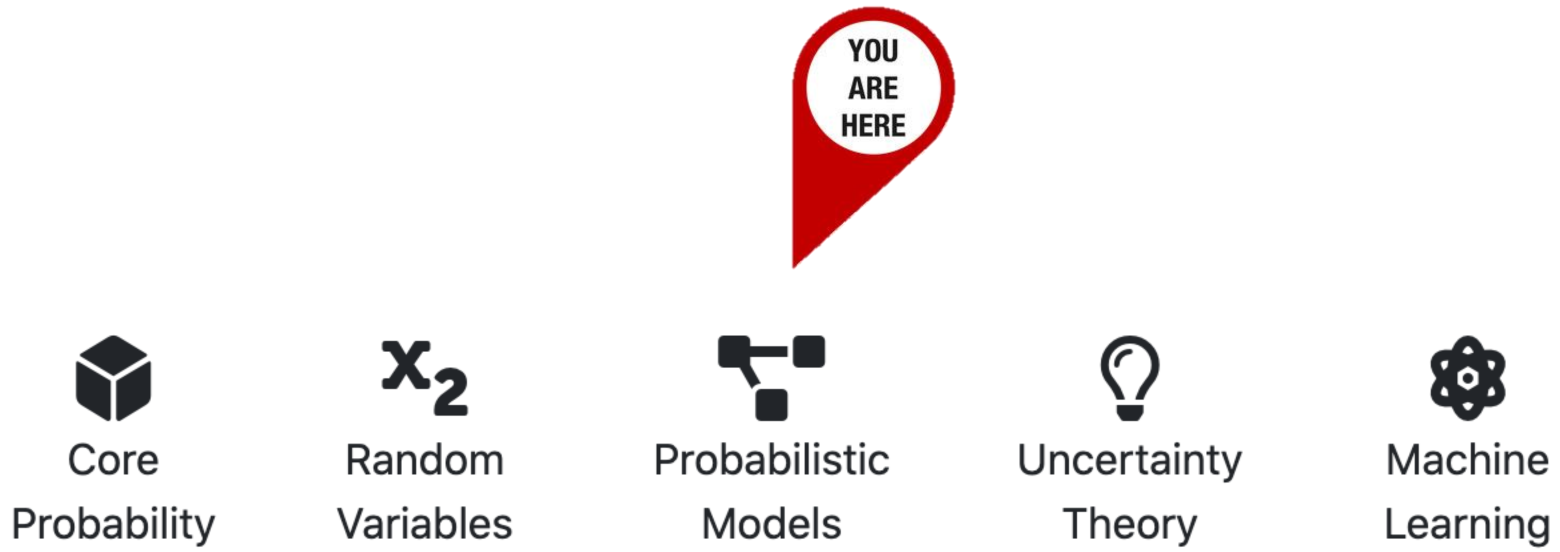
$$= \frac{f(X=10)}{f(X=5)}$$

$$= \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}}$$

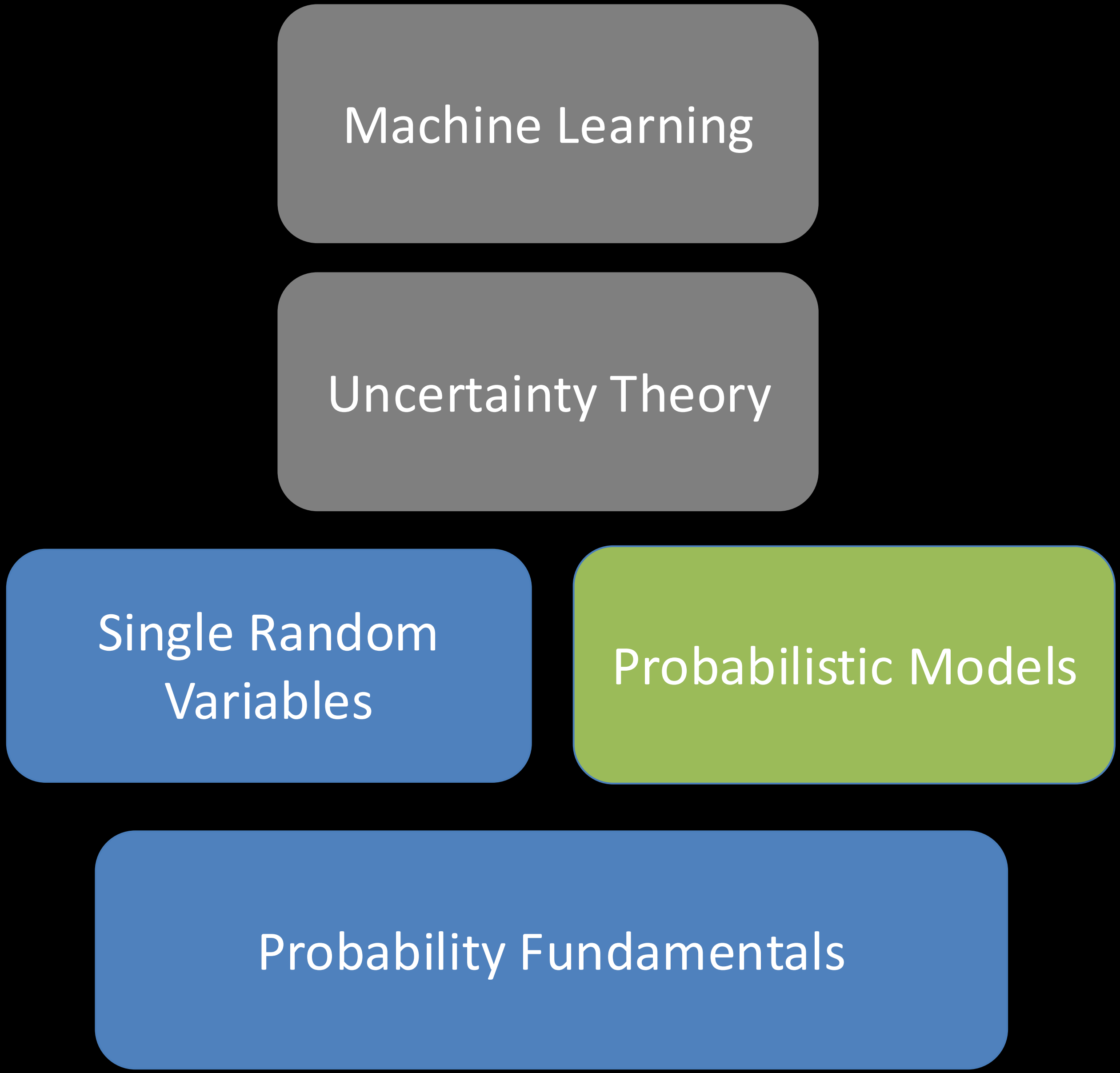
$$= \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{(10-10)^2}{4}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(5-10)^2}{4}}}$$

$$= \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

# Where are we in CS109?



# CS109



Machine Learning
Uncertainty Theory
Single Random Variables
Probabilistic Models
Probability Fundamentals

[suspense]

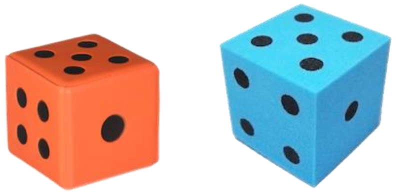
# Discrete Probabilistic Models

# The world is full of interesting probability problems

Have multiple random variables interacting with one another

# Joint probability mass functions

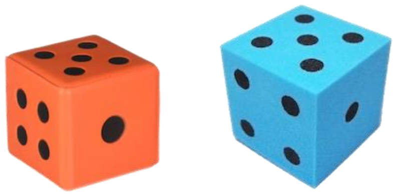
Roll two 6-sided dice, yielding values $X$ and $Y$.



| $X$ | $P(X = 1)$ | $P(X = k)$ |
|---|---|---|
| random variable | probability of an event | probability mass function |

# Joint probability mass functions

Roll two 6-sided dice, yielding values $X$ and $Y$.



$X$

random variable

$P(X = 1)$

probability of an event

$P(X = k)$

probability mass function

$X, Y$

random variables

$P(X = 1 \text{ and } Y = 6)$

$P(X = 1, Y = 6)$

recall: the comma

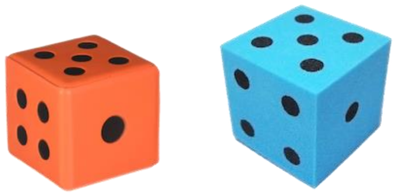
probability of the intersection of two events

$P(X = a, Y = b)$

joint probability mass function

# Two dice

Roll two 6-sided dice, yielding values $X$ and $Y$.



1. What is the joint PMF of $X$ and $Y$?

$$P(X = a, Y = b) = 1/36 \qquad (a, b) \in \{(1,1), \dots, (6,6)\}$$

| | $X$ = 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Y$ = 1 | 1/36 | … | … | … | … | 1/36 |
| 2 | … | … | … | … | | |
| 3 | … | … | … | … | … | … |
| 4 | … | … | … | … | … | … |
| 5 | … | … | … | … | … | … |
| 6 | 1/36 | … | … | … | … | 1/36 |

$P(X = 4, Y = 3)$

**Probability table**

- All possible outcomes for several discrete RVs
- Not parametric (e.g., parameter $p$ in Ber($p$))

# Dating at Stanford. Data from a few years ago

| | Single | In a relationship | It's complicated |
|---|---|---|---|
| Freshman | 0.13 | 0.08 | 0.02 |
| Sophomore | 0.17 | 0.11 | 0.02 |
| Junior | 0.09 | 0.10 | 0.02 |
| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

# Joint is Complete Information!

| | Single | Relationship | Complicated |
|---|---|---|---|
| Frosh | 0.13 | 0.08 | 0.02 |
| Soph | 0.17 | 0.11 | 0.02 |
| Junior | 0.09 | 0.10 | 0.02 |
| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

A joint distribution is complete information. It can be used to answer any probability question.

# Joint table: mutually exclusive and covers sample space.

| | Single | Relationship | Complicated |
|---|---|---|---|
| Frosh | 0.13 | 0.08 | 0.02 |
| Soph | 0.17 | 0.11 | 0.02 |
| Junior | 0.09 | 0.10 | 0.02 |
| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

Each combination is mutually exclusive, and they span the sample space

$$\sum_{x \in X} \sum_{y \in Y} P(x, y) = 1$$

X is dating status.
Y is year.

# Joint table: mutually exclusive and covers sample space.

| | Single | Relationship | Complicated |
|---|---|---|---|
| Frosh | 0.13 | 0.08 | 0.02 |
| Soph | 0.17 | 0.11 | 0.02 |
| Junior | 0.09 | ? | 0.02 |
| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

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$$\sum_{x \in X} \sum_{y \in Y} P(x, y) = 1$$

X is dating status.
Y is year.

# Joint table: mutually exclusive and covers sample space.

| | Single | Relationship | Complicated |
|---|---|---|---|
| Frosh | $0.07 \cdot k$ | $0.04 \cdot k$ | $0.01 \cdot k$ |
| Soph | $0.09 \cdot k$ | $0.05 \cdot k$ | $0.01 \cdot k$ |
| Junior | $0.05 \cdot k$ | $0.05 \cdot k$ | $0.01 \cdot k$ |
| Senior | $0.01 \cdot k$ | $0.03 \cdot k$ | $0.01 \cdot k$ |
| 5+ | $0.03 \cdot k$ | $0.03 \cdot k$ | $0.02 \cdot k$ |

Each combination is mutually exclusive, and they span the sample space

$$\sum_{x \in X} \sum_{y \in Y} P(x, y) = 1$$

X is dating status.
Y is year.

$$\sum_{x \in \text{table}} k \cdot x = 1 \qquad k \cdot \sum_{x \in \text{table}} x = 1 \qquad k = \frac{1}{\sum_{x \in \text{table}} x}$$

# Joint table: mutually exclusive and covers sample space.

| | Single | Relationship | Complicated |
|---|---|---|---|
| Frosh | 0.13 | 0.08 | 0.02 |
| Soph | 0.17 | 0.11 | 0.02 |
| Junior | 0.09 | 0.10 | 0.02 |
| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

Each combination is mutually exclusive, and they span the sample space

$$\sum_{x \in X} \sum_{y \in Y} P(x, y) = 1$$

X is dating status.
Y is year.

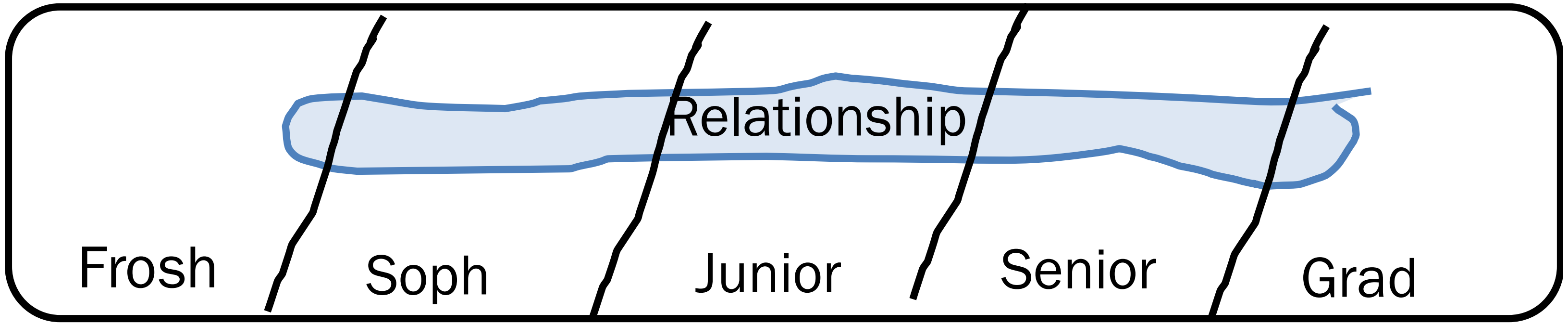
# What is the probability someone is in a relationship?

| | Single | Relationship | Complicated |
|---|---|---|---|
| Frosh | 0.13 | 0.08 | 0.02 |
| Soph | 0.17 | 0.11 | 0.02 |
| Junior | 0.09 | 0.10 | 0.02 |
| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

We can use the law of total probability!

X is dating status. Y is year.

$$P(X = \text{relation}) = \sum_{y \in Y} P(X = \text{relation}, Y = y)$$



# What is the probability someone is in a relationship?

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| Senior | 0.02 | 0.07 | 0.01 |
| 5+ | 0.06 | 0.09 | 0.04 |

We can use the law of total probability!
X is dating status. Y is year.

$$P(X = \text{relation}) = \sum_{y \in Y} P(X = \text{relation}, Y = y)$$

$$P(X = \text{single}) = \sum_{y \in Y} P(X = \text{single}, Y = y)$$

$$P(Y = \text{frosh}) = \sum_{x \in X} P(X = x, Y = \text{frosh})$$

$$P(Y = \text{soph}) = \sum_{x \in X} P(X = x, Y = \text{soph})$$

Welcome the marginal

# Marginal Distribution

For two discrete joint random variables $X$ and $Y$,
the **joint probability mass function** is defined as:

$$P(X = a, Y = b)$$

The **marginal distributions** of the joint PMF are defined as:

$$P(X = a) = \sum_y P(X = a, Y = y)$$

$$P(Y = b) = \sum_x P(X = x, Y = b)$$

Use marginal distributions to get a 1-D RV from a joint PMF.

Why is that called the marginal?

# Quick note on independence

# If $A$ and $B$ are independent:

Joint

$$P(A = a, B = b) = P(A = a) \cdot P(B = b)$$

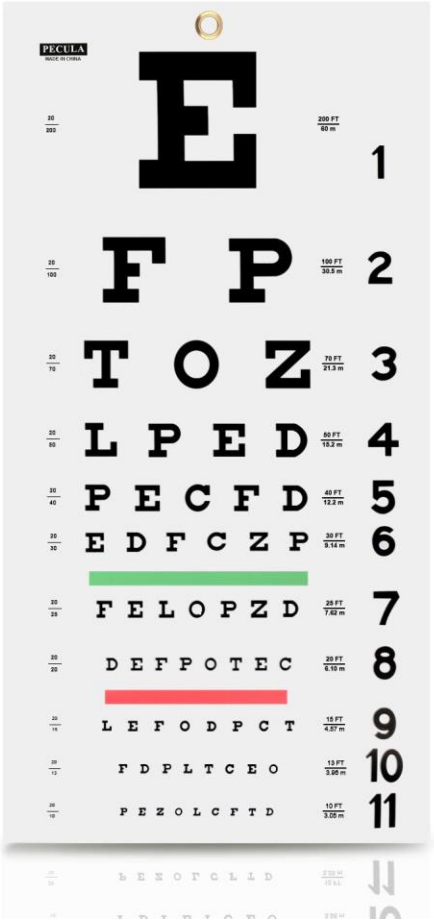
$$P(A < a, B < b) = P(A < a) \cdot P(B < b)$$

For all a, b

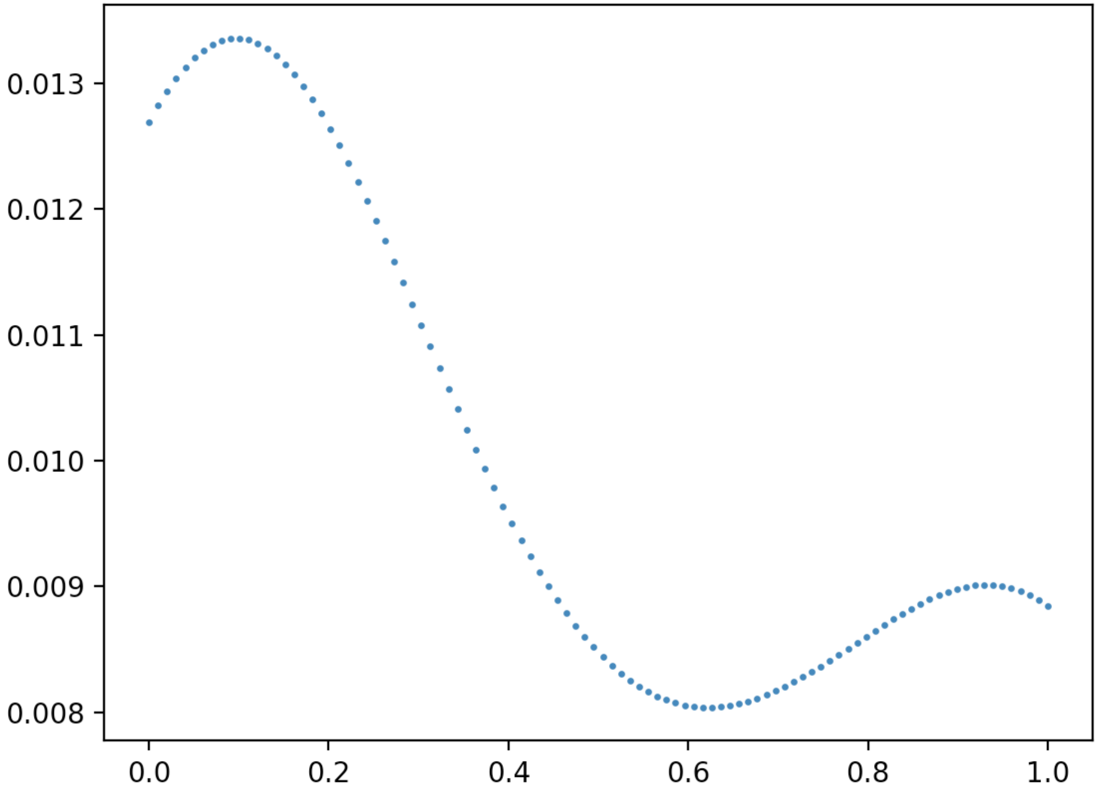
Get Pumped…

# Biggest Game Changer this half of CS109

# Belief in **Vision** Given **User Responses**

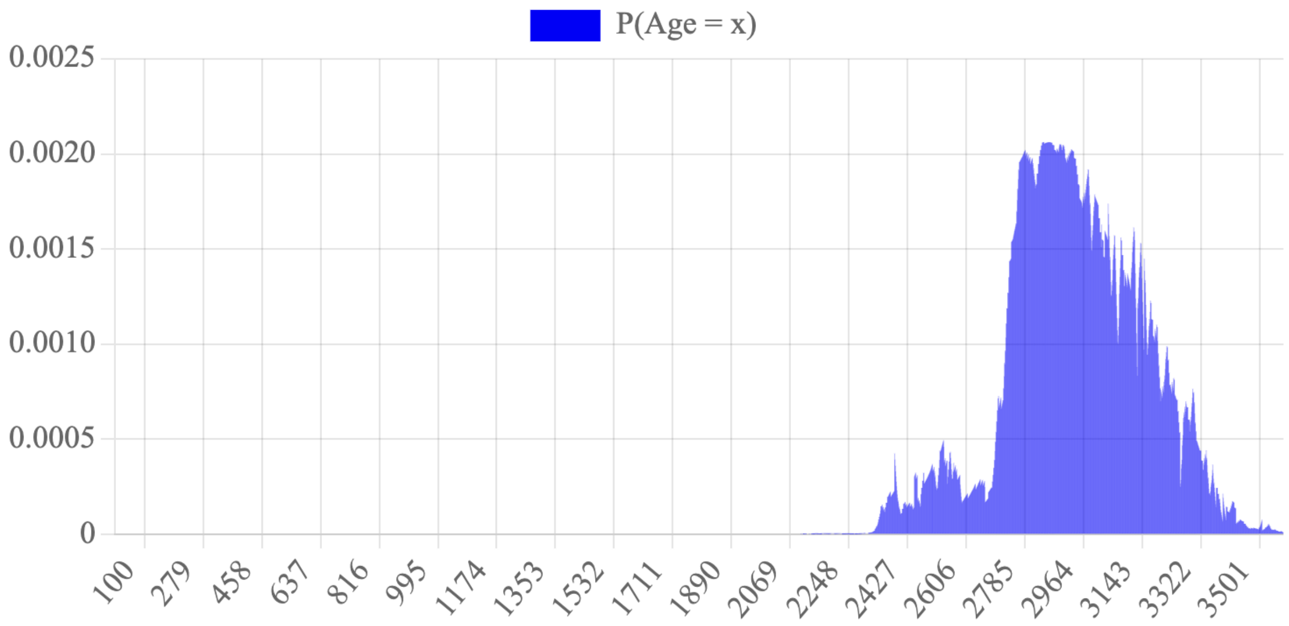



P(Ability to See | Observed Responses)



# Belief in **Age** Given **Observed C14**

P(Age| Observed C14)



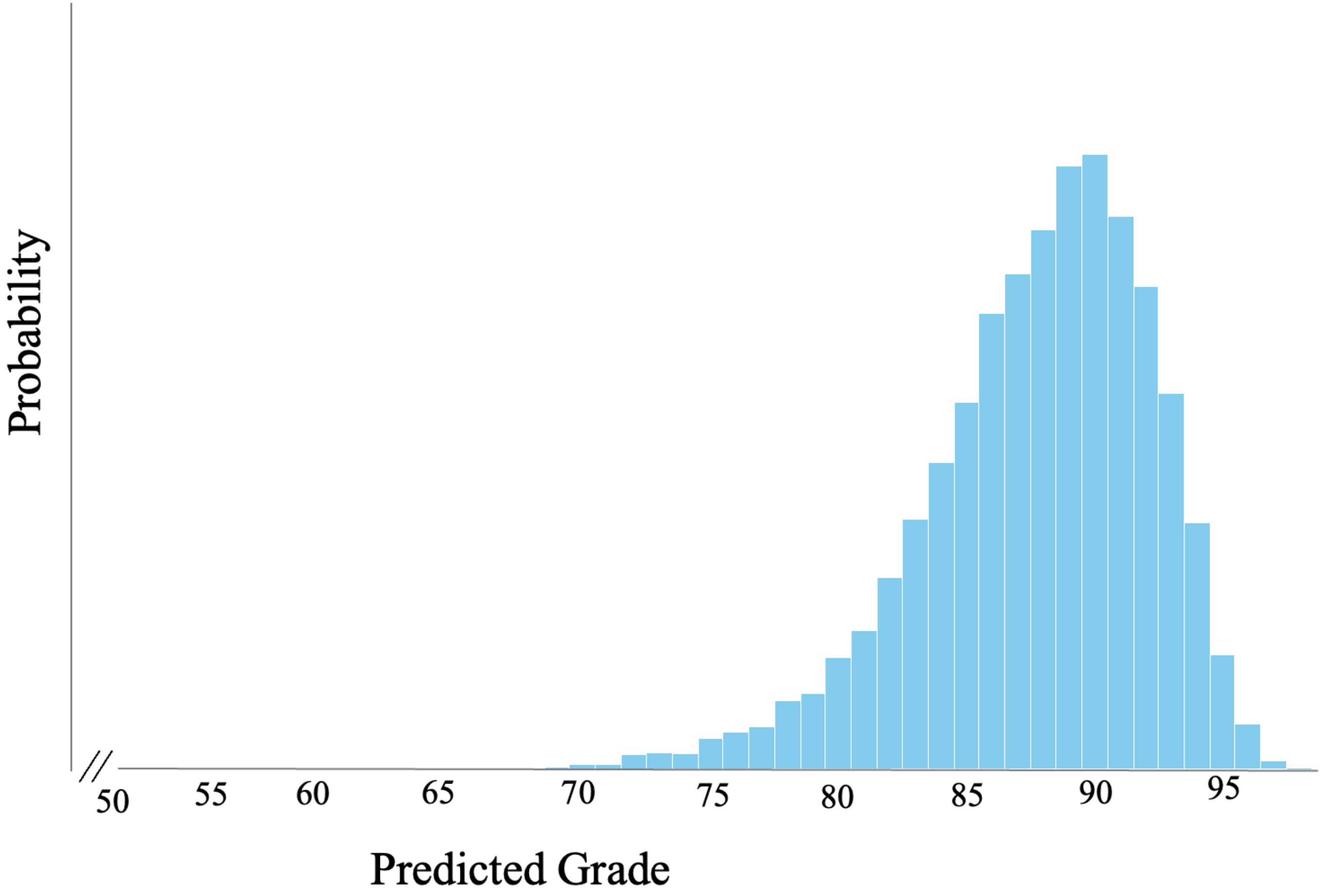
# Belief in **Course Grade** Given **Assignment Scores So Far**

Assns Scores So Far

Score on Pset 1

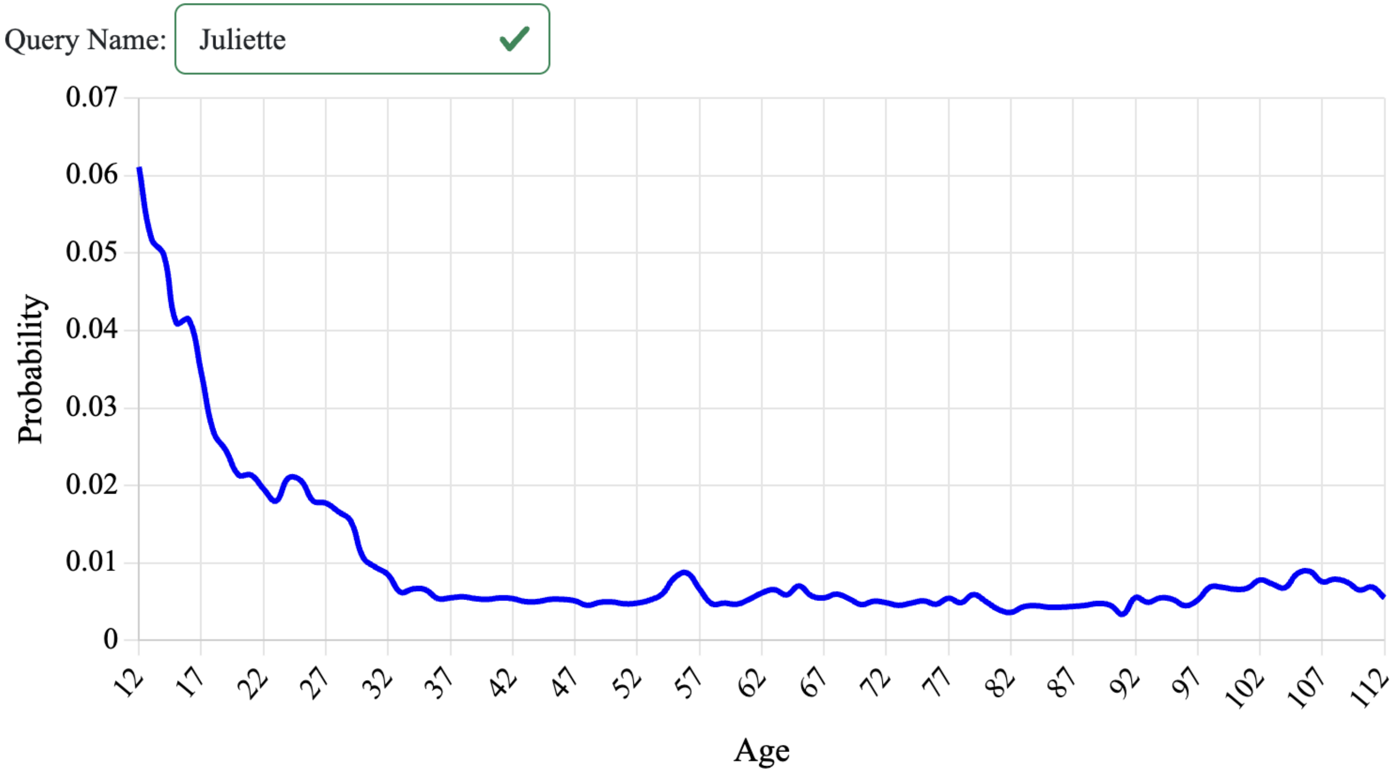
Score on Pset 2

P(Final Grade | Assns Scores So Far)

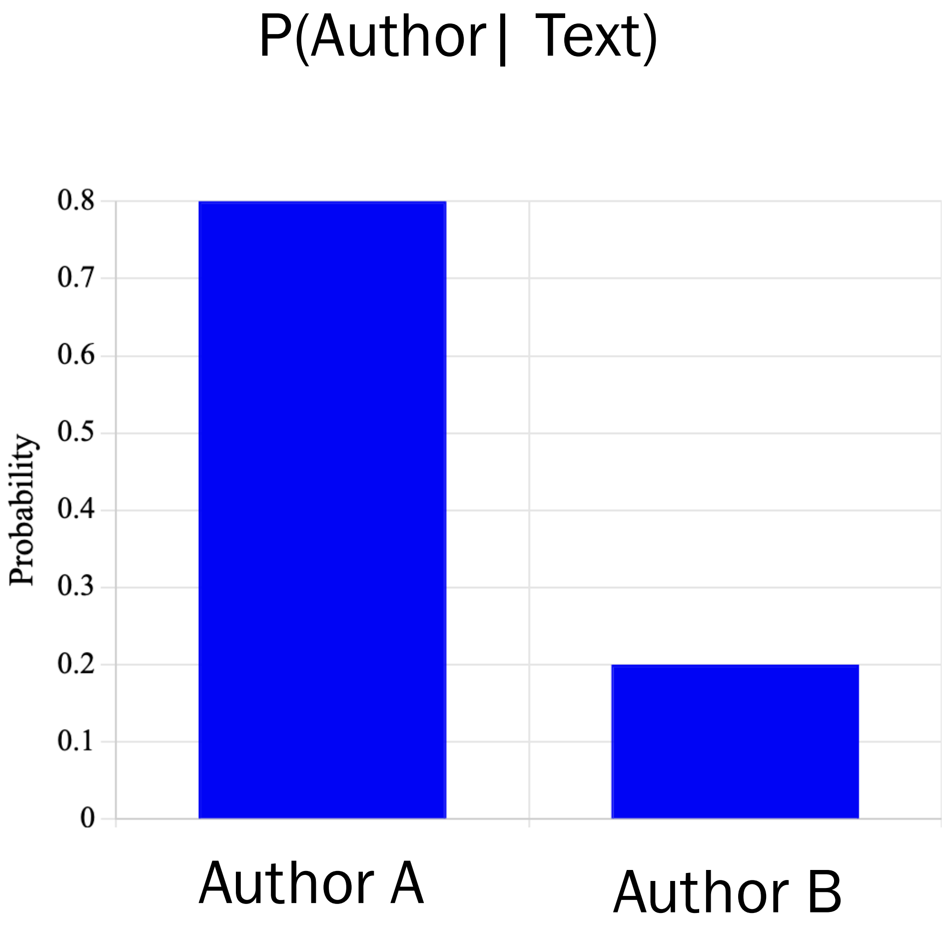


# Belief in **Age** Given **Name**


P(Age | Name)



# Belief in **Author** Given **Text**



# Today: Inference

**Inference** *noun*

Updating one's belief about a random variable (or multiple) based on conditional knowledge regarding another random variable (or multiple) in a probabilistic model.

TLDR: conditional probability with random variables.

# Bayes Theorem

Likelihood of evidence

Posterior belief

Prior belief

$$P(B|E) = \frac{P(E|B) \quad P(B)}{P(E)}$$

Normalization constant

# Bayes with Discrete Random Variables

Let M be a **discrete** random variable

Let N be a **discrete** random variable

$$P(M = 2 | N = 3) = \frac{P(N = 3 | M = 2) P(M = 2)}{P(N = 3)}$$

More generally

$$P(M = m | N = n) = \frac{P(N = n | M = m) P(M = m)}{P(N = n)}$$

Shorthand notation

$$P(m|n) = \frac{P(n|m) P(m)}{P(n)}$$

# I Heard That

Let X be the **change in gaze** (measured in degrees) over 3 seconds after a sound is played

| Value of $X$ | PMF of $X$ given Baby can hear the sound | PMF of $X$ given Baby can **not** hear the sound |
|---|---|---|
| 0 to 5 | 0.08 | 0.40 |
| 5 to 10 | 0.15 | 0.30 |
| 10 to 15 | 0.35 | 0.12 |
| 15 to 20 | 0.20 | 0.08 |
| 20 to 25 | 0.12 | 0.05 |
| Above 25 | 0.10 | 0.05 |

$$\text{P(can hear the sound)} = \frac{3}{4}$$

You observe $X = 0$. What is the probability the baby **can** hear the sound?

# Question: Have I Been Given the Joint?

Let X be the **change in gaze** (measured in degrees) over 3 seconds after a sound is played

| Value of $X$ | PMF of $X$ given Baby can hear the sound | PMF of $X$ given Baby can **not** hear the sound |
|---|---|---|
| 0 to 5 | 0.08 | 0.40 |
| 5 to 10 | 0.15 | 0.30 |
| 10 to 15 | 0.35 | 0.12 |
| 15 to 20 | 0.20 | 0.08 |
| 20 to 25 | 0.12 | 0.05 |
| Above 25 | 0.10 | 0.05 |

$$\text{P(can hear the sound)} = \frac{3}{4}$$

You observe $X = 0$. What is the probability the baby **can** hear the sound?

# I Heard That

| Value of $X$ | PMF of $X$ given Baby can hear the sound | PMF of $X$ given Baby can **not** hear the sound |
|---|---|---|
| 0 to 5 | 0.08 | 0.40 |
| 5 to 10 | 0.15 | 0.30 |
| 10 to 15 | 0.35 | 0.12 |
| 15 to 20 | 0.20 | 0.08 |
| 20 to 25 | 0.12 | 0.05 |
| Above 25 | 0.10 | 0.05 |

$$\text{P(can hear the sound)} = \frac{3}{4}$$

You observe $X = 0$. What is the probability the baby **can** hear the sound?

Y = 1 means the child can hear the sound

$$P(Y = 1|X = 0) = \frac{P(X = 0 \mid Y = 1)\, P(Y = 1)}{P(X = 0 \mid Y = 1)\, P(Y = 1) + P(X = 0 \mid Y = 0)\, P(Y = 0)}$$

$$P(Y = 1|X = 0) = \frac{0.08 * 0.75}{0.08 * 0.75 + 0.40 * 0.25} = \frac{3}{8}$$

# I Heard That with Continuous

**Normal Assumption**:

For babies who can hear sounds, we approximate their gaze movement after the sound is played as: $N(\mu = 15, \sigma^2 = 25)$.

For babies who can **not hear** sounds, we approximate gaze movement as $N(\mu = 8, \sigma^2 = 25)$.

For a new baby we observe a 14 degree movement after the sound is played. What is your belief that a baby can hear, under The Normal Assumption?

How do you handle observing a continuous value?

# Aside

# All the Bayes Belong to Us

**M,N are discrete. X, Y are continuous**

OG Bayes

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

Mix Bayes #1

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

Mix Bayes #2

$$P(n|x) = \frac{f(x|n)P(n)}{f(x)}$$

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

TLDR:

If random variable is discrete: use PMF

If random variable is continuous: use PDF

# Mixing Discrete and Continuous

Let X be a continuous random variable

Let N be a discrete random variable

$$P(N = n | X = x) = \frac{P(X = x | N = n) P(N = n)}{P(X = x)}$$

$$P(N = n | X = x) = \frac{f(X = x | N = n) \cdot \epsilon \cdot P(N = n)}{f(X = x) \cdot \epsilon}$$

$$P(N = n | X = x) = \frac{f(X = x | N = n) \cdot P(N = n)}{f(X = x)}$$

# Mixing Discrete and Continuous

Let X be a **continuous** random variable

Let N be a **discrete** random variable

$$P(X = x | N = n) = \frac{P(N = n | X = x) P(X = x)}{P(N = n)}$$

Change notation

$$P(x|n) = \frac{P(n|x)P(x)}{P(n)}$$

$$f(x|n) \cdot \epsilon_x = \frac{P(n|x)f(x) \cdot \epsilon_x}{P(n)}$$

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

# All the Bayes Belong to Us

**M,N are discrete. X, Y are continuous**

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$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

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$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

Mix Bayes #2

$$P(n|x) = \frac{f(x|n)P(n)}{f(x)}$$

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

TLDR:

If random variable is discrete: use PMF

If random variable is continuous: use PDF

# LOTP? Chain Rule? You can play too!

**N is discrete**. **X is continuous**

Chain Rule

$$f(N = n, X = x) = f(X = x|N = n)P(N = n)$$

Law of total probability

$$f(X = x) = \sum_n f(X = x|N = n)P(N = n)$$

# End Aside

# I Heard That with Continuous

**Normal Assumption**:

For babies who can hear sounds, we approximate their gaze movement after the sound is played as: $N(\mu = 15, \sigma^2 = 25)$.

For babies who can **not** hear sounds, we approximate gaze movement as $N(\mu = 8, \sigma^2 = 25)$.

For a new baby we observe a 14 degree movement after the sound is played. What is your belief that a baby can hear, under The Normal Assumption?

# Equivalently

**Normal Assumption**: For babies who can hear sounds, we approximate their gaze movement after the sound is played as: $N(\mu = 15, \sigma^2 = 25)$.

For babies who can **not** hear sounds, we approximate gaze movement as $N(\mu = 8, \sigma^2 = 25)$.

For a new baby we observe a 14 degree movement after the sound is played. What is your belief that a baby can hear, under The Normal Assumption?

```
def sample ():
  # bernoulli sample
  can_hear = rand_bern(0.75)
  if can_hear == 1:
     # gaussian sample
     return rand_gauss (mu = 15 , std = 5)
  else:
     # gaussian sample
     return rand_gauss (mu = 8, std = 5)
```

The function sample returned the value 14. What is the probability that can_hear was 1?

# All the Bayes Belong to Us

**M,N are discrete**. **X, Y are continuous**

OG Bayes

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

Mix Bayes #1

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

Mix Bayes #2

$$P(n|x) = \frac{f(x|n)P(n)}{f(x)}$$

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

TLDR:

If random variable is discrete: use PMF

If random variable is continuous: use PDF

# Pedagogical Pause

# Goal: Inference

Change your belief
***distribution***
(Joint, PMF, or PDF)
of random variables,
based on
observations

*Note in the earlier examples, we were updating Bernoulli Random Variables

# Lets Play Number of Function!

# Number or Function?

$$P(X = 2 | Y = 5)$$

Number

# Number or Function?

$$P(X = x | Y = 2)$$

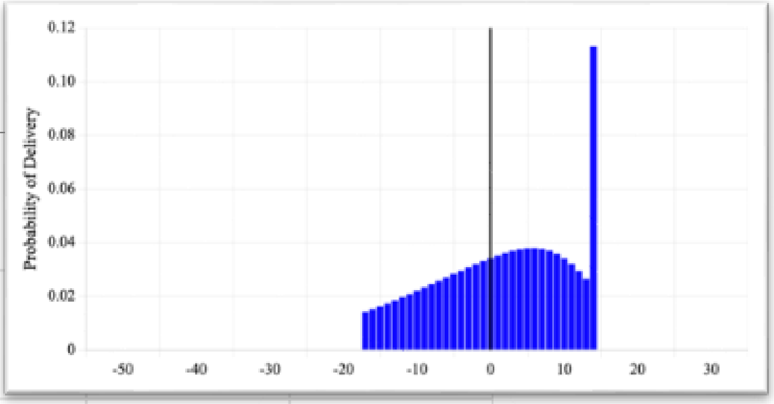
Function

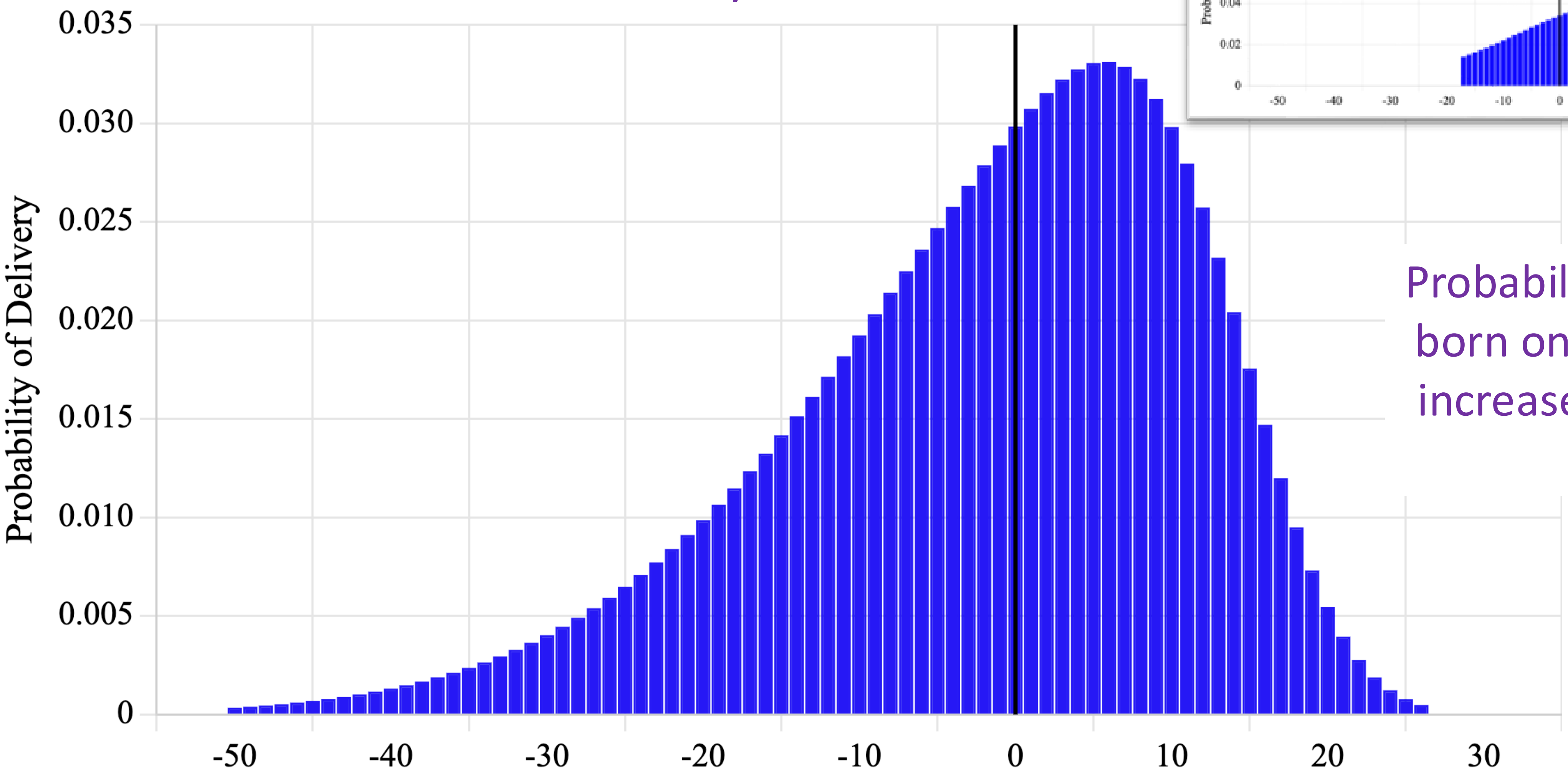
(a probability mass function
if discrete)

# Baby Delivery Timing


# Baby Delivery

Its 17 days before the due date and still no baby



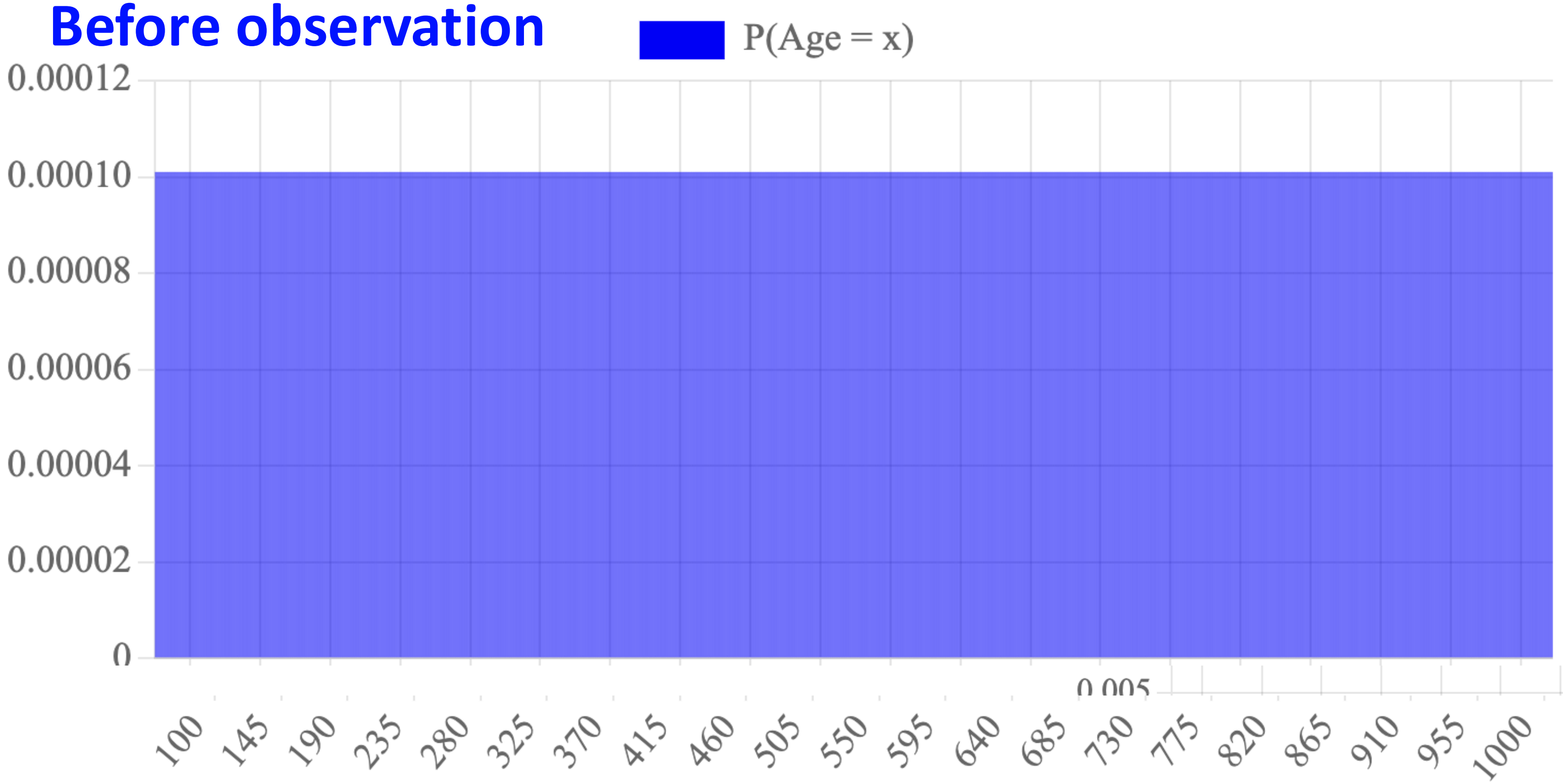

Probability baby is born on due date increases by 13%

# Bayesian Carbon Dating

1. Take a fixed size sample from a dead thing
2. Count C14 in Sample
3. Know the probability distribution for when it died
4. Profit

# Bayesian Carbon Dating

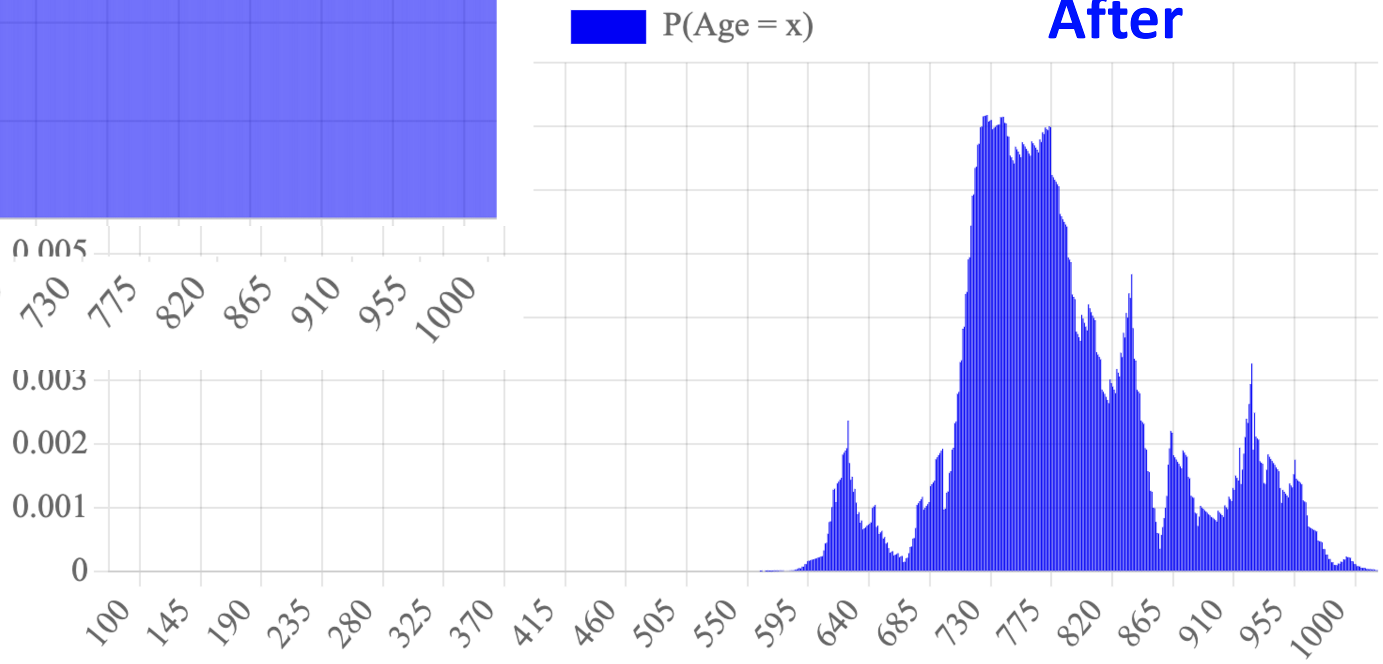
## Before observation




## Observation

Remaining C14: 900

## After



# Warmup with Code

$$P(A=i|M=900) = P(M=900|A=i) \cdot P(A=i) \cdot K$$

```
def update_belief(m = 900):
  """
  Returns a dictionary A, where A[i] contains the
  corresponding probability, P(A = i| M = 900).
  m is the number of C14 molecules remaining and i
  is age in years. i is in the range 100 to 10000
  """
  pr_A = {}
  n_years = 9901
  for i in range(100,10000+1):
    prior = 1 / n_years # P(A = i)
    likelihood = calc_likelihood(m, i) # P(M=m | A=i)
    pr_A[i] = prior * likelihood
  # implicitly computes the normalization constant
  normalize(pr_A)
  return pr_A
```

# Normalize in Python

```
# list normalization
def normalize_list(data_list):
    total_sum = np.sum(data_list)
    return np.array(data_list) / total_sum

>>> norm = normalize_list([10, 20, 30, 40])
>>> np.sum(norm) # 1.0, always (within floating point error)
```

return
[0.1 0.2 0.3 0.4]

---

```
# dictionary normalization
def normalize_dict(data_dict):
    total_sum = sum(data_dict.values())
    normalized = {}
    for key, value in data_dict.items():
        normalized[key] = value / total_sum
    return normalized

>>> norm = normalize_dict({'a': 100, 'b': 200, 'c': 300})
>>> np.sum(norm.values()) # 1.0, always (within floating point error)
```

return
{'a': 0.166, 'b': 0.333, 'c': 0.5}

# Bayesian Carbon Dating: Inference Overview

Let A be how many years old the sample is (A = 100 means the sample is 100 years old)
Let M be the observed amount of C14 left in the sample

$$P(A = i | M = 900) = \frac{P(M = 900 | A = i) P(A = i)}{P(M = 900)}$$

$$= P(M = 900 | A = i) \cdot P(A = i) \cdot K$$

Such that

$$K = \frac{1}{\sum_j P(M = 900 | A = ij) P(A = j)}$$

# Understanding why Denom. is a Constant

$A$ = age
$M$ = measured C14

$$P(A=i|M=900) = P(M=900|A=i) \cdot P(A=i) \cdot \frac{1}{P(M=900)}$$

Notice which term doesn't change as i changes (four example calculations).

$$P(A=100|M=900) = P(M=900|A=100) \cdot P(A=100) \cdot \frac{1}{P(M=900)}$$

$$P(A=200|M=900) = P(M=900|A=200) \cdot P(A=200) \cdot \frac{1}{P(M=900)}$$

$$P(A=300|M=900) = P(M=900|A=300) \cdot P(A=300) \cdot \frac{1}{P(M=900)}$$

$$P(A=400|M=900) = P(M=900|A=400) \cdot P(A=400) \cdot \frac{1}{P(M=900)}$$

Doesn't change

Changes with *i*

# Bayesian Carbon Dating: Inference Overview

Let A be how many years old the sample is (A = 100 means the sample is 100 years old)
Let M be the observed amount of C14 left in the sample

$$P(A = i|M = 900) = \frac{P(M = 900|A = i)P(A = i)}{P(M = 900)}$$

$$= P(M = 900|A = i) \cdot P(A = i) \cdot K$$

Such that

$$K = \frac{1}{\sum_j P(M = 900|A = ij)P(A = j)}$$

# Understanding Through Code

$$P(A = i | M = 900) = P(M = 900 | A = i) \cdot P(A = i) \cdot K$$

```
def update_belief(m = 900):
  """
  Returns a dictionary A, where A[i] contains the
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  """
  pr_A = {}
  n_years = 9901
  for i in range(100,10000+1):
    prior = 1 / n_years # P(A = i)
    likelihood = calc_likelihood(m, i) # P(M=m | A=i)
    pr_A[i] = prior * likelihood
  # implicitly computes the normalization constant
  normalize(pr_A)
  return pr_A
```

# Carbon Dating Likelihood Math

# Probability of Having 900 Remain

$$P(M = 900 | A = i)$$

There were originally 1000 C14 molecules.
Each molecule remains independently with equal probability $p_i$
What is the probability that 900 remain?

$$M \sim \text{Bin}(n = 1000, p = p_i)$$

$$P(M = 900 | A = i) = \binom{1000}{900} (p_i)^{900} \cdot (1 - p_i)^{100}$$

Each molecules' time to live is exponential with λ = 1/8267
Let T be the time to decay for any one molecule

$$T \sim \text{Exp}(\lambda = 1/8267) \quad p_i = P(T > i) = 1 - P(T < i) = e^{-\frac{i}{8267}}$$

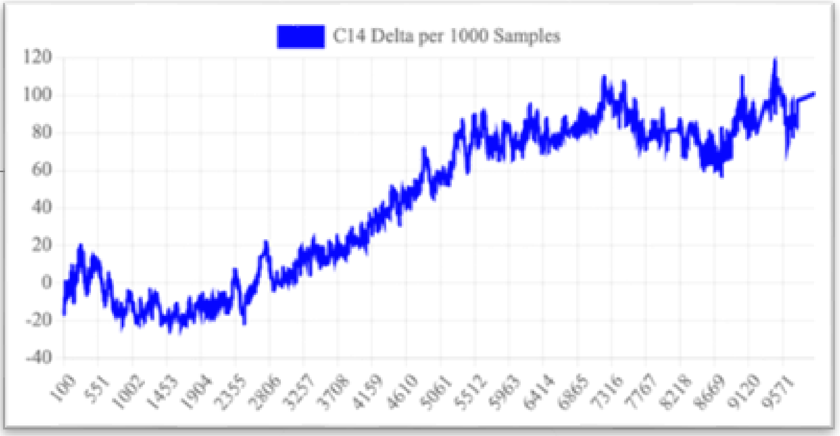
# Probability of Having 900 Remain

$$P(M = 900 | A = i)$$

$$\binom{1000}{900} (e^{\frac{-i}{8267}})^{900} \cdot (1 - (e^{\frac{-i}{8267}}))^{100}$$

```
def calc_likelihood(m = 900, age):
  """
  Computes P(M = m | A = age), the probability of
  having m molecules left given the sample is age
  years old. Uses the exponential decay of C14
  """
  n_original = 1000
  p_remain = math.exp(-age/C14_MEAN_LIFE)
  return stats.binom.pmf(m, n_original, p_remain)
```

# Probability of Having 900 Remain

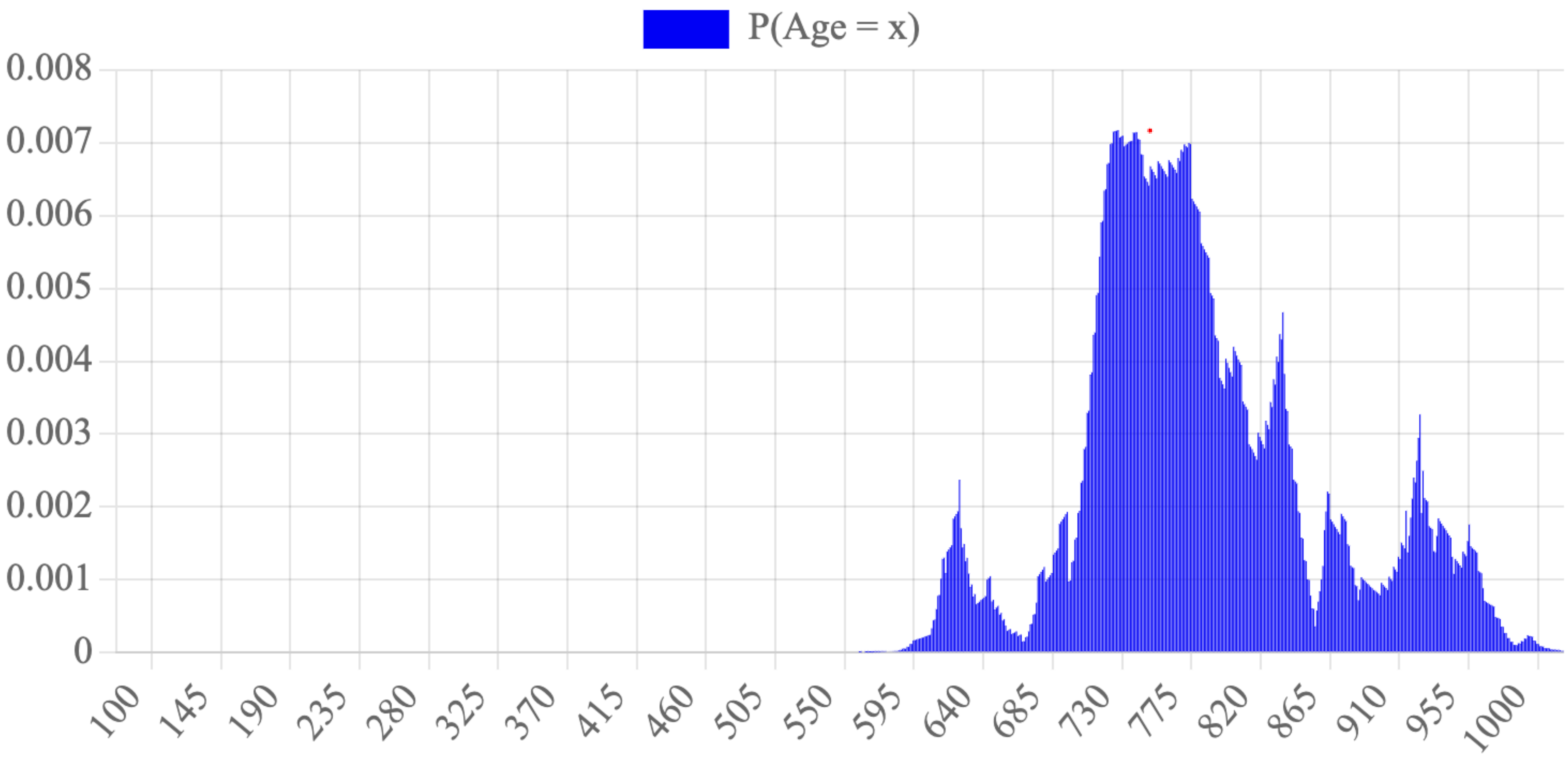



$$P(M = 900 | A = i)$$

```
def calc_likelihood(m = 900, age):
  """
  Computes P(M = m | A = age), the probability of
  having m molecules left given the sample is age
  years old. Uses the exponential decay of C14
  """
  n_original = 1000 + delta_start(age)
  p_remain = math.exp(-age/C14_MEAN_LIFE)
  return stats.binom.pmf(m, n_original, p_remain)
```

# Posterior Belief in Age

Remaining C14: 900



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