11: Joint (Multivariate) Distributions

Lisa Yan
April 29, 2020
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Normal Approximation
Normal RVs

\[ X \sim \mathcal{N}(\mu, \sigma^2) \]

• Used to model many real-life situations because it maximizes entropy (i.e., randomness) for a given mean and variance

• Also useful for approximating the Binomial random variable!
Website testing

- 100 people are given a new website design.
- \( X = \# \text{ people whose time on site increases} \)
- The design actually has no effect, so \( P(\text{time on site increases}) = 0.5 \) independently.
- CEO will endorse the new design if \( X \geq 65 \).

What is \( P(\text{CEO endorses change})? \) Give a numerical approximation.

Approach 1: Binomial

Define

\[ X \sim \text{Bin}(n = 100, p = 0.5) \]

Want: \( P(X \geq 65) \)

Solve

\[
P(X \geq 65) = \sum_{i=65}^{100} \binom{100}{i} 0.5^i (1 - 0.5)^{100-i}
\]
Don’t worry, Normal approximates Binomial

(We’ll explain why in 2 weeks’ time)
Website testing

- 100 people are given a new website design.
- \( X = \# \text{ people whose time on site increases} \)
- The design actually has no effect, so \( P(\text{time on site increases}) = 0.5 \) independently.
- CEO will endorse the new design if \( X \geq 65 \).

What is \( P(\text{CEO endorses change})? \) *Give a numerical approximation.*

**Approach 1: Binomial**

Define
\[
X \sim \text{Bin}(n = 100, p = 0.5)
\]

Want: \( P(X \geq 65) \)

Solve
\[
P(X \geq 65) \approx 0.0018
\]

⚠ ⚠ (this approach is actually missing something)

**Approach 2: approximate with Normal**

Define
\[
Y \sim \mathcal{N}(\mu, \sigma^2)
\]

\[
\mu = np = 50
\]
\[
\sigma^2 = np(1 - p) = 25
\]
\[
\sigma = \sqrt{25} = 5
\]

Solve
\[
P(X \geq 65) \approx P(Y \geq 65) = 1 - F_Y(65)
\]
\[
= 1 - \Phi\left(\frac{65 - 50}{5}\right) = 1 - \Phi(3) \approx 0.0013
\]

(this approach is actually missing something)
Website testing (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100, 0.5)$.

$$P(X \geq 65) \approx P(Y \geq 64.5)$$

$$\approx 0.0018$$

You must perform a continuity correction when approximating a Binomial RV with a Normal RV.
Continuity correction

If $Y \sim \mathcal{N}(np, np(1 - p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

<table>
<thead>
<tr>
<th>Discrete (e.g., Binomial) probability question</th>
<th>Continuous (Normal) probability question</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = 6)$</td>
<td>$P(X &lt; 6)$</td>
</tr>
<tr>
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<td>$P(X &gt; 6)$</td>
</tr>
<tr>
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Continuity correction

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<td>$P(5.5 \leq Y \leq 6.5)$</td>
</tr>
<tr>
<td>$P(X \geq 6)$</td>
<td>$P(Y \geq 5.5)$</td>
</tr>
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</tr>
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<td>$P(Y \leq 6.5)$</td>
</tr>
</tbody>
</table>
Who gets to approximate?

\[ X \sim \text{Bin}(n, p) \]
\[ E[X] = np \]
\[ \text{Var}(X) = np(1 - p) \]

\[ Y \sim \text{Poi} (\lambda) \]
\[ \lambda = np \]

\[ Y \sim \mathcal{N} (\mu, \sigma^2) \]
\[ \mu = np \]
\[ \sigma^2 = np(1 - p) \]
Who gets to approximate?

Poisson approximation

- $n$ large ($> 20$), $p$ small ($< 0.05$)
- slight dependence okay

Normal approximation

- $n$ large ($> 20$), $p$ mid-ranged ($np(1-p) > 10$)
- independence

1. If there is a choice, use Normal to approximate.
2. When using Normal to approximate a discrete RV, use a continuity correction.
Discrete Joint RVs
From last time

What is the probability that the Warriors win?
How do you model zero-sum games?

\[ P(A_W > A_B) \]

This is a probability of an event involving two random variables!
Joint probability mass functions

Roll two 6-sided dice, yielding values $X$ and $Y$.

$X$  
random variable

$P(X = 1)$  
probability of an event

$P(X = k)$  
probability mass function
Joint probability mass functions

Roll two 6-sided dice, yielding values $X$ and $Y$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X = 1)$</th>
<th>$P(X = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>random variable</td>
<td>probability of an event</td>
<td>probability mass function</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X$, $Y$</th>
<th>$P(X = 1 \cap Y = 6)$</th>
<th>$P(X = a, Y = b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>random variables</td>
<td>$P(X = 1, Y = 6)$</td>
<td>joint probability mass function</td>
</tr>
<tr>
<td>new notation: the comma</td>
<td>probability of the intersection of two events</td>
<td></td>
</tr>
</tbody>
</table>
Discrete joint distributions

For two discrete joint random variables $X$ and $Y$, the joint probability mass function is defined as:

$$p_{X,Y}(a, b) = P(X = a, Y = b)$$

The marginal distributions of the joint PMF are defined as:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

Use marginal distributions to get a 1-D RV from a joint PMF.
Two dice

Roll two 6-sided dice, yielding values $X$ and $Y$.

1. What is the joint PMF of $X$ and $Y$?

$$p_{X,Y}(a, b) = \begin{cases} 1/36 & (a, b) \in \{(1,1), \ldots, (6,6)\} \\ \text{otherwise} \end{cases}$$

<table>
<thead>
<tr>
<th>$Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/36</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>1/36</td>
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<tr>
<td>2</td>
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</tr>
<tr>
<td>6</td>
<td>1/36</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>1/36</td>
</tr>
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</table>

Probability table

- All possible outcomes for several discrete RVs
- Not parametric (e.g., parameter $p$ in $\text{Ber}(p)$)
Two dice

Roll two 6-sided dice, yielding values $X$ and $Y$.

1. What is the joint PMF of $X$ and $Y$?

   \[ p_{X,Y}(a, b) = 1/36 \quad (a, b) \in \{(1,1), \ldots, (6,6)\} \]

2. What is the marginal PMF of $X$?

   \[ p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6} \quad a \in \{1, \ldots, 6\} \]

   \[ P(X=1) = P(X=1, Y=1) + \ldots + P(X=1, Y=6) \]
A computer (or three) in every house.

Consider households in Silicon Valley.

• A household has $X$ Macs and $Y$ PCs.
• Each house has a maximum of 3 computers (Macs + PCs) in the house.

1. What is $P(X = 1, Y = 0)$, the missing entry in the probability table?

<table>
<thead>
<tr>
<th></th>
<th>$X$ (# Macs)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$ (# PCs)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>.16</td>
<td>?</td>
<td>.07</td>
<td>.04</td>
</tr>
<tr>
<td>1</td>
<td>.12</td>
<td>.14</td>
<td>.12</td>
<td>0</td>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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A joint PMF must sum to 1:

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$
A computer (or three) in every house.

Consider households in Silicon Valley.
- A household has $X$ Macs and $Y$ PCs.
- Each house has a maximum of 3 computers (Macs + PCs) in the house.

2. How do you compute the marginal PMF of $X$?

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<tr>
<td>0</td>
<td>.39</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>38</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>04</td>
<td></td>
<td></td>
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- A: sum rows here
- B: sum cols here
- C: sum cols here
A computer (or three) in every house.

Consider households in Silicon Valley.
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- Each house has a maximum of 3 computers (Macs + PCs) in the house.

2. How do you compute the marginal PMF of $X$?

- **A.** $p_{X,Y}(x,0) = P(X = x, Y = 0)$
- **B.** Marginal PMF of $X$ $p_X(x) = \sum_y p_{X,Y}(x,y)$
- **C.** Marginal PMF of $Y$ $p_Y(y) = \sum_x p_{X,Y}(x,y)$

To find a marginal distribution over one variable, sum over all other variables in the joint PMF.
A computer (or three) in every house.

Consider households in Silicon Valley.
- A household has $X$ Macs and $Y$ PCs.
- Each house has a maximum of 3 computers (Macs + PCs) in the house.

3. Let $C = X + Y$. What is $P(C = 3)$?

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3. Let $C = X + Y$. What is $P(C = 3)$?

\[
\begin{array}{c|cccc}
X \text{ (# Macs)} & 0 & 1 & 2 & 3 \\
\hline
0 & .16 & .12 & .07 & .04 \\
1 & .12 & .14 & .12 & 0 \\
2 & .07 & .12 & 0 & 0 \\
3 & .04 & 0 & 0 & 0 \\
\end{array}
\]

\[
P(C = 3) = P(X + Y = 3) = \sum_{x} \sum_{y} P(X + Y = 3 | X = x, Y = y)P(X = x, Y = y)
\]

\[
= P(X = 0, Y = 3) + P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 3, Y = 0)
\]

\[
= .04 + .12 + .07 + .04 = 0.32
\]

We’ll come back to sums of RVs next lecture!
Multinomial RV
Recall the good times

Permutations

\[ n! \]

How many ways are there to order \( n \) objects?
Counting unordered objects

Binomial coefficient

How many ways are there to group \( n \) objects into two groups of size \( k \) and \( n - k \), respectively?

\[
\binom{n}{k} = \frac{n!}{k! (n - k)!}
\]

Called the binomial coefficient because of something from Algebra

Multinomial coefficient

How many ways are there to group \( n \) objects into \( r \) groups of sizes \( n_1, n_2, ..., n_r \), respectively?

\[
\binom{n}{n_1, n_2, ..., n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}
\]

Multinomials generalize Binomials for counting.
Probability

**Binomial RV**

What is the probability of getting \( k \) successes and \( n - k \) failures in \( n \) trials?

\[
P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}
\]

**Multinomial RV**

What is the probability of getting \( c_1 \) of outcome 1, \( c_2 \) of outcome 2, ..., and \( c_m \) of outcome \( m \) in \( n \) trials?

Multinomial RVs also generalize Binomial RVs for probability!
Multinomial Random Variable

Consider an experiment of $n$ independent trials:
- Each trial results in one of $m$ outcomes. $P(\text{outcome } i) = p_i, \sum_{i=1}^{m} p_i = 1$
- Let $X_i = \#$ trials with outcome $i$

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

where $\sum_{i=1}^{m} c_i = n$ and $\sum_{i=1}^{m} p_i = 1$

- Multinomial # of ways of ordering the outcomes
- Probability of each ordering is equal + mutually exclusive
Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.
What is the probability of getting:

• 1 one
• 1 two
• 0 threes
• 2 fours
• 0 fives
• 3 sixes
Hello dice rolls, my old friends

A 6-sided die is rolled 7 times. What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

\[
P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3) = \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7
\]
Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.
What is the probability of getting:

- 1 one
- 0 threes
- 0 fives
- 1 two
- 2 fours
- 3 sixes

\[ P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3) \]

\[ = \binom{7}{1,1,0,2,0,3} \left( \frac{1}{6} \right)^1 \left( \frac{1}{6} \right)^1 \left( \frac{1}{6} \right)^0 \left( \frac{1}{6} \right)^2 \left( \frac{1}{6} \right)^0 \left( \frac{1}{6} \right)^3 = 420 \left( \frac{1}{6} \right)^7 \]

Choose where the sixes appear
Probability of rolling a six this many times
11: Joint (Multivariate) Distributions

Lisa Yan
April 29, 2020
Normal RVs

\[ X \sim \mathcal{N}(\mu, \sigma^2) \]

- Used to model many real-life situations because it maximizes entropy (i.e., randomness) for a given mean and variance.
- Also useful for approximating the Binomial random variable!
Who gets to approximate?

\[ X \sim \text{Bin}(n, p) \]
\[ E[X] = np \]
\[ \text{Var}(X) = np(1 - p) \]

\[ Y \sim \text{Poi}(\lambda) \]
\[ \lambda = np \]

\[ Y \sim \mathcal{N}(\mu, \sigma^2) \]
\[ \mu = np \]
\[ \sigma^2 = np(1 - p) \]

- Computing probabilities on Binomial RVs is often computationally expensive.
- Two reasonable approximations, but when to use which?

\[ n \text{ large (> 20)} \]
\[ p \text{ small (< 0.05)} \]
slight dependence okay

\[ n \text{ large (> 20)}, p \text{ mid-ranged (np}(1-p) > 10) \]

independence

need continuity correction
Check out the question on the next slide (Slide 38). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/46501
Stanford Admissions (a while back)

Stanford accepts 2480 students.
- Each accepted student has $68\%$ chance of attending (independent trials)
- Let $X = \#$ of students who will attend

What is $P(X > 1745)$? Give a numerical approximation.

Strategy:  
A. Just Binomial  
B. Poisson  
C. Normal  
D. None/other
Stanford Admissions (a while back)

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What is $P(X > 1745)$? Give a numerical approximation.

Strategy:

A. Just Binomial
   - not an approximation (also computationally expensive)
B. Poisson
   - $p = 0.68$, not small enough
C. Normal
   - Variance $np(1 - p) = 540 > 10$
D. None/other

Define an approximation

Let $Y \sim \mathcal{N}(E[X], \text{Var}(X))$

$E[X] = np = 1686 \approx \mu$
$\text{Var}(X) = np(1 - p) \approx 540 \rightarrow \sigma = 23.3$

$P(X > 1745) \approx P(Y \geq 1745.5)$

Continuity correction

$P(Y \geq 1745.5) = 1 - F(1745.5)$

$= 1 - \Phi \left( \frac{1745.5 - 1686}{23.3} \right)$

$= 1 - \Phi(2.54) \approx 0.0055$
Changes in Stanford Admissions

Stanford accepts 2480 students.
• Each accepted student has 68% chance of attending (independent trials)
• Let $X = \#$ of students who will attend

What is $P(X > 1745)$? Give a numerical approximation.

Overview for the Class of 2022

- Total Applicants: 47,451
- Total Admits: 2,071
- Total Enrolled: 1,706

Admit rate: 4.3%
Yield rate: 81.9%

People love coming to Stanford!
Multinomial Random Variable

Consider an experiment of \( n \) independent trials:

- Each trial results in one of \( m \) outcomes. \( P(\text{outcome } i) = p_i, \ \sum_{i=1}^{m} p_i = 1 \)
- Let \( X_i = \# \) trials with outcome \( i \)

Joint PMF

\[
P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \ldots p_m^{c_m}
\]

where \( \sum_{i=1}^{m} c_i = n \) and \( \sum_{i=1}^{m} p_i = 1 \)

Example:

- Rolling 2 twos, 3 threes, and 5 fives on 10 rolls of a fair-sided die
- Generating a random 5-word phrase with 1 “the”, 2 “bacon”, 1 “put”, 1 “on”
Hello dice rolls, my old friends

A 6-sided die is rolled 7 times. What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

\[ P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3) \]

\[ = \binom{7}{3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7 \]
Parameters of a Multinomial RV?

$X \sim \text{Bin}(n, p)$ has parameters $n, p$...

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$p$: probability of success outcome on a single trial

A Multinomial RV has parameters $n, p_1, p_2, \ldots, p_m$ (Note $p_m = 1 - \sum_{i=1}^{m-1} p_i$)

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

$p_i$: probability of outcome $i$ on a single trial

Where do we get $p_i$ from?
Interlude for jokes/announcements
Announcements

Quiz #1
Time frame: Thursday 4/30 12:00am-11:59pm PT
Covers: Up to end of Week 3 (including Lecture 9)
Info and practice: https://web.stanford.edu/class/cs109/exams/quizzes.html

Thoughts pre-quiz:
• A checkpoint for you, not other people
• We are all here to learn. This exam was designed for a range of students.
• Typesetting will take a bit of time (total: ~2 hr + typeset)
• Take breaks, stretch, sleep
• The staff and I are here for you.

Other things this week
• Section optional (not graded), attend any section
• Friday’s concept check #12 EC
Estimating Coronavirus Prevalence by Cross-Checking Countries

"We’ll make the modeling assumption that \( N_{ij} \) is a Poisson distribution with rate parameter \( A_{ij} \* \lambda_i \* \alpha_j \). What this means is that the expected number of cases should be equal to the total amount of travel, times some source-dependent multiplier \( \alpha_j \) ..., times some country-dependent multiplier \( \lambda_i \) (the infection prevalence in country i)."

https://medium.com/@jsteinhardt/estimating-coronavirus-prevalence-by-cross-checking-countries-c7e4211f0e18

CS109 Current Events Spreadsheet
The Federalist Papers
Probabilistic text analysis

Ignoring the order of words...

What is the probability of any given word that you write in English?

- $P(\text{word} = \text{“the”}) > P(\text{word} = \text{“pokemon”})$
- $P(\text{word} = \text{“Stanford”}) > P(\text{word} = \text{“Cal”})$

Probabilities of *counts* of words = Multinomial distribution

A document is a large multinomial.

(according to the Global Language Monitor, there are 988,968 words in the English language used on the internet.)
Probabilistic text analysis

Probabilities of *counts* of words = Multinomial distribution

Example document:

“When my late husband was alive he deposited some amount of Money with china Bank in which the amount will be declared to you once you respond to this message indicating your interest in helping to receive the fund and use it for Gods work as my wish.”

\[
P\left(\text{spam} \mid \frac{n!}{1!1!1!1!\cdots3!}p_{\text{bank}}p_{\text{fund}}\cdots p_{\text{to}}\right)
\]

Note: \( P\left(\text{bank} \mid \text{spam writer}=\right) \gg P\left(\text{bank} \mid \text{writer= you}\right) \)
Old and New Analysis

Authorship of the Federalist Papers

• 85 essays advocating ratification of the US constitution
• Written under the pseudonym “Publius” (really, Alexander Hamilton, James Madison, John Jay)

Who wrote which essays?

• Analyze probability of words in each essay and compare against word distributions from known writings of three authors

Let’s write a program!
Probabilistic text analysis

Probabilities of counts of words = Multinomial distribution

What about probability of those same words in someone else’s writing?

• $P\left( \text{word} = \text{“probability”} \mid \text{writer} = \text{you} \right) > P\left( \text{word} = \text{“probability”} \mid \text{writer} = \text{non-CS109 student} \right)$

To determine authorship:

1. Estimate $P(\text{word} \mid \text{writer})$ from known writings
2. Use Bayes’ Theorem to determine $P(\text{writer} \mid \text{document})$ for a new writing!

Who wrote the Federalist Papers?
Step 1. Generate probability lookups

\[ \text{Frequency} \]
\[ \text{Congress in Madison} = \frac{\text{# times Congress appears}}{\text{# total words in Madison.txt}} \]

\[ \text{P (Congress in Madison)} = \frac{\text{freq} \cdot k}{\# \text{total words in Madison.txt} + \# \text{words in English}} \]

Sample space: English dictionary

freq: stored in MadisonWordProb

\[ \text{P (Congress \& freq)} \]
Step 1. Generate probability lookups

\( m_i \) Frequency of word \( i \) in Madison's writing, \( \propto P(\text{word } i \mid \text{Madison}) \)

\( h_i \) Frequency of word \( i \) in Hamilton's writing, \( \propto P(\text{word } i \mid \text{Hamilton}) \)

4. How will these values help us compute probabilities on a sentence being written by Hamilton or Madison?
   - "The People The Congress"
   - "People Congress The Rambutans"

5. [reach] Why don't the total numbers for just Madison add up to *exactly* one?

6. [reach] How does returning EPSILON for unknown words help us?
Step 1. Generate probability lookups

\[ P(\text{"The Congress The People" | Madison}) \]
\[ = P(\text{counts of words | Madison}) \]
\[ = \binom{4}{2,1,1,1} M_{\text{Congress}}^1 M_{\text{people}}^2 M_{\text{state}}^0 M_{\text{1}}^0 M_{\text{2}}^0 \]

\[ P(\text{"People Congress The Rambutans" | Hamilton}) \]
\[ = \binom{4}{1,1,1,1,1} h_{\text{11111}} h_{\text{long}} h_{\text{the}} \]
\[ = 10^{-6} P(\text{rambutan | Hamilton}) \]
Step 2. Unknown document counts

2. How would you represent the probability of Madison writing this document with a Multinomial? Let $c_i$ be the count of word $i$.

\[
\begin{align*}
\left( \lambda = 2170 \right) & \quad c_1 \quad c_2 \quad \cdots \\
(c_1, c_2, \ldots, c_m) & \quad M_1 \quad M_2 \quad \cdots \\
\text{\# words} & \quad \text{unique} \\
\end{align*}
\]
Step 3. Bayes’ Theorem

\[ P(\text{Madison} | \text{unknownDoc}) = \frac{P(\text{unknownDoc} | \text{Madison})P(\text{Madison})}{P(\text{unknownDoc})} \quad \text{(Bayes)} \]

Assume that \( P(\text{writer}) = 0.5 \). We can rewrite this into a decision:

\[ \frac{P(\text{unknownDoc} | \text{Madison})}{P(\text{unknownDoc} | \text{Hamilton})} > 1 \]

\( \iff \)

\[ \frac{P(M | D)}{P(H | D)} > 1 \]

\[ \Rightarrow \quad \frac{P(M | D)P(D)}{P(H | D)P(D)} > 1 \]

\[ \Rightarrow \quad \frac{P(D | M)}{P(D | H)} > 1 \]

If true, report Madison.
Step 3. Bayes’ Theorem

Assume that $P(\text{writer}) = 0.5$. We can rewrite this into a decision:

$$\frac{P(\text{unknownDoc}|\text{Madison})}{P(\text{unknownDoc}|\text{Hamilton})} > 1$$

(If true, Madison is writer)