Convolution
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What happens when you add random variables?
Let $X$ and $Y$ be independent random variables
- $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$
- $X + Y \sim \text{Bin}(n_1 + n_2, p)$

Intuition:
- $X$ has $n_1$ trials and $Y$ has $n_2$ trials
  - Each trial has same “success” probability $p$
- Define $Z$ to be $n_1 + n_2$ trials, each with success prob. $p$
- $Z \sim \text{Bin}(n_1 + n_2, p)$, and also $Z = X + Y$
If only it were always that simple
What is the probability that $X + Y = n$?

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>k</th>
<th>P(X = X, Y = Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>n</td>
<td>0</td>
<td>$P(X = 0, Y = n)$</td>
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<tr>
<td>1</td>
<td>n - 1</td>
<td>1</td>
<td>$P(X = 1, Y = n-1)$</td>
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<tr>
<td>2</td>
<td>n - 2</td>
<td>2</td>
<td>$P(X = 2, Y = n-2)$</td>
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<tr>
<td>n</td>
<td>0</td>
<td>n</td>
<td>$P(X = n, Y = 0)$</td>
</tr>
</tbody>
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The Insight to Convolution Proofs

What is the probability that \( X + Y = n \)?

\[
P(X + Y = n) = \sum_{k=0}^{n} P(X = k, Y = n - k)
\]

Since this is the OR or mutually exclusive events

\[
P(X + Y = n) = \sum_{k=0}^{n} P(X = k)P(Y = n - k)
\]

If the random variables are independent
Sum of Independent Poissons

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Recall the Binomial Theorem
Let $X$ and $Y$ be independent random variables
- $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$
- $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$

Proof: (just for reference)
- Rewrite $(X + Y = n)$ as $(X = k, Y = n - k)$ where $0 \leq k \leq n$

$$P(X + Y = n) = \sum_{k=0}^{n} P(X = k, Y = n - k) = \sum_{k=0}^{n} P(X = k)P(Y = n - k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} = e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} = e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k}$$

- Noting Binomial theorem: $(\lambda_1 + \lambda_2)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k}$

- $P(X + Y = n) = \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n$ so, $X + Y = n \sim \text{Poi}(\lambda_1 + \lambda_2)$
Reference: Sum of Independent RVs

- Let $X$ and $Y$ be independent Binomial RVs
  - $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$
  - $X + Y \sim \text{Bin}(n_1 + n_2, p)$
  - More generally, let $X_i \sim \text{Bin}(n_i, p)$ for $1 \leq i \leq N$, then
    $$\left( \sum_{i=1}^{N} X_i \right) \sim \text{Bin}\left( \sum_{i=1}^{N} n_i, p \right)$$

- Let $X$ and $Y$ be independent Poisson RVs
  - $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$
  - $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$
  - More generally, let $X_i \sim \text{Poi}(\lambda_i)$ for $1 \leq i \leq N$, then
    $$\left( \sum_{i=1}^{N} X_i \right) \sim \text{Poi}\left( \sum_{i=1}^{N} \lambda_i \right)$$
We talked about sum of Binomial and Poisson…who’s missing from this party? Uniform.
Summation: not just for the 1%
Let $X$ and $Y$ be independent random variables.

- Probability Density Function (PDF) of $X + Y$:

$$f_{X+Y}(a) = \int_{y=-\infty}^{\infty} f_X(a-y) f_Y(y) \, dy$$

- In discrete case, replace $\int$ with $\sum$, and $f(y)$ with $p(y)$
Integration with Constraint

\[\iiint_{x^2 + y^2 < 1} f_{x,y} \, dy \, dx = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{x,y} \, dy \, dx\]
Let $X$ and $Y$ be independent random variables.

- Cumulative Distribution Function (CDF) of $X + Y$:

$$F_{X+Y}(a) = P(X + Y \leq a)$$

$$= \int_{x+y\leq a} \int f_X(x) f_Y(y) \, dx \, dy$$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_X(x) f_Y(y) \, dx \, dy$$

$$= \int_{y=-\infty}^{\infty} F_X(a-y) f_Y(y) \, dy$$

- In discrete case, replace $\int$ with $\sum$, and $f(y)$ with $p(y)$.
Sum of Independent Uniforms

- Let $X$ and $Y$ be independent random variables
  - $X \sim \text{Uni}(0, 1)$ and $Y \sim \text{Uni}(0, 1) \Rightarrow f(x) = 1$ for $0 \leq x \leq 1$

For both $X$ and $Y$
Let $X$ and $Y$ be independent random variables
- $X \sim \text{Uni}(0, 1)$ and $Y \sim \text{Uni}(0, 1) \Rightarrow f(x) = 1$ for $0 \leq x \leq 1$
- What is PDF of $X + Y$?

$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a - y) f_Y(y) \, dy = \int_{y=0}^{1} f_X(a - y) \, dy$$

When $a = 0.5$:

$$f_{X+Y}(0.5) = \int_{y=?}^{0.5} f_X(0.5 - y) \, dy$$

$$= \int_{0}^{0.5} f_X(0.5 - y) \, dy$$

$$= \int_{0}^{0.5} 1 \, dy$$

$$= 0.5$$

**Sum of Independent Uniforms**

**Graph**: A graph showing the probability density function $f_{X+Y}(a)$ for $a = 0.5$. The graph is a line segment from $(0, 0)$ to $(1, 1)$, indicating a uniform distribution from 0 to 1.
Let X and Y be independent random variables

- $X \sim \text{Uni}(0, 1)$ and $Y \sim \text{Uni}(0, 1) \Rightarrow f(x) = 1$ for $0 \leq x \leq 1$
- What is PDF of $X + Y$?

$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y) f_Y(y) \, dy = \int_{y=0}^{1} f_X(a-y) \, dy$$

When $a = 1.5$:

$$f_{X+Y}(1.5) = \int_{y=0}^{1.5} f_X(1.5 - y) \, dy$$

$$= \int_{0.5}^{1} f_X(1.5 - y) \, dy$$

$$= \int_{0.5}^{1} 1 \, dy$$

$$= 0.5$$
Let $X$ and $Y$ be independent random variables

- $X \sim \text{Uni}(0, 1)$ and $Y \sim \text{Uni}(0, 1) \Rightarrow f(x) = 1$ for $0 \leq x \leq 1$

- What is PDF of $X + Y$?

$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y) f_Y(y) \, dy = \int_{y=0}^{1} f_X(a-y) \, dy$$

When $a = 1$:

$$f_{X+Y}(1) = \int_{y=0}^{1} f_X(1-y) \, dy$$

$$= \int_{0}^{1} f_X(1-y) \, dy$$

$$= \int_{0}^{1} 1 \, dy$$

$$= 1$$
Sum of Independent Uniforms

- Let $X$ and $Y$ be independent random variables
  - $X \sim \text{Uni}(0, 1)$ and $Y \sim \text{Uni}(0, 1) \Rightarrow f(x) = 1$ for $0 \leq x \leq 1$
  - What is PDF of $X + Y$?
    $$f_{X+Y}(a) = \int_{y=0}^{1} f_x(a-y) f_y(y) \, dy = \int_{y=0}^{1} f_x(a-y) \, dy$$
  - When $0 \leq a \leq 1$ and $0 \leq y \leq a$, $0 \leq a-y \leq 1 \Rightarrow f_x(a-y) = 1$
    $$f_{X+Y}(a) = \int_{y=0}^{a} dy = a$$
  - When $1 \leq a \leq 2$ and $a-1 \leq y \leq 1$, $0 \leq a-y \leq 1 \Rightarrow f_x(a-y) = 1$
    $$f_{X+Y}(a) = \int_{y=a-1}^{1} dy = 2-a$$
- Combining:
  $$f_{X+Y}(a) = \begin{cases} 
a & 0 \leq a \leq 1 \\
2-a & 1 < a \leq 2 \\
0 & \text{otherwise}
\end{cases}$$
Let $X$ and $Y$ be independent random variables

- $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
- $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Generally, have $n$ independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$:

$$\left( \sum_{i=1}^{n} X_i \right) \sim N\left( \sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2 \right)$$
Virus Infections

- Say you are working with the WHO to plan a response to the initial conditions of a virus:
  - Two exposed groups
  - P1: 50 people, each independently infected with $p = 0.1$
  - P2: 100 people, each independently infected with $p = 0.4$
  - Question: Probability of more than 40 infections?

Sanity check: Should we use the Binomial Sum-of-RVs shortcut?
A. YES!
B. NO!
C. Other/none/more
• Say you are working with the WHO to plan a response to the initial conditions of a virus:
  - Two exposed groups
  - P1: 50 people, each independently infected with $p = 0.1$
  - P2: 100 people, each independently infected with $p = 0.4$
  - $A = \#$ infected in P1
    - $A \sim \text{Bin}(50, 0.1) \approx X \sim \text{N}(5, 4.5)$
  - $B = \#$ infected in P2
    - $B \sim \text{Bin}(100, 0.4) \approx Y \sim \text{N}(40, 24)$
  - What is $P(\geq 40$ people infected$)$?
  - $P(A + B \geq 40) \approx P(X + Y \geq 39.5)$
  - $X + Y = W \sim \text{N}(5 + 40 = 45, 4.5 + 24 = 28.5)$

\[
P(W \geq 39.5) = P\left(\frac{W - 45}{\sqrt{28.5}} > \frac{39.5 - 45}{\sqrt{28.5}}\right) = 1 - \Phi(-1.03) \approx 0.8485\]
Linear Transform

\[ X \sim N(\mu, \sigma^2) \]

\[ Y = X + X = 2 \cdot X \]

\[ Y \sim N(2\mu, 4\sigma^2) \]

\[ Y = X + X = 2 \cdot X \]

\[ X + X \sim N(\mu + \mu, \sigma^2 + \sigma^2) \]

\[ Y \sim N(2\mu, 2\sigma^2) \]

\( X \) is not independent of \( X \)
End sum of independent vars