16: Continuous Joint Distributions

Lisa Yan
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Continuous joint distributions
Remember target?

Good times...

\[
\frac{59}{309} = 0.191
\]
Quick check: What is the probability that a dart hits at (456.2344132343, 532.1865739012)?

The CS109 logo was created by throwing 500,000 darts according to a joint distribution.

If we throw another dart according to the same distribution, what is $P(\text{dart hits within } r \text{ pixels of center})$?
P(dart hits within $r$ pixels of center)?

Possible dart counts (in 100x100 boxes)
Possible dart counts (in 50x50 boxes)
Possible dart counts (in infinitesimally small boxes)
Continuous joint probability density functions

If two random variables $X$ and $Y$ are jointly continuous, then there exists a joint probability density function $f_{X,Y}$ defined over $-\infty < x, y < \infty$ such that:

$$P(a_1 \leq X \leq a_2, \ b_1 \leq Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y) dy \ dx$$
From one continuous RV to jointly continuous RVs

Single continuous RV $X$
- PDF $f_X$ such that $\int_{-\infty}^{\infty} f_X(x) \, dx = 1$
- Integrate to get probabilities

Jointly continuous RVs $X$ and $Y$
- PDF $f_{X,Y}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1$
- Double integrate to get probabilities

Probability for jointly continuous RVs is **volume** under a surface.
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.
- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

Write down the definite double integral that must integrate to 1:
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

Write down the definite double integral that must integrate to 1:

$$\int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy = 1 \quad \text{or} \quad \int_{x=0}^{1} \int_{y=0}^{2} xy \, dy \, dx = 1$$

(used in next slide)
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

0. Set up integral:

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, dx \, dy = \int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy$$

1. Evaluate inside integral by treating $y$ as a constant:

$$\int_{y=0}^{2} \left( \int_{x=0}^{1} xy \, dx \right) \, dy = \int_{y=0}^{2} y \left( \int_{x=0}^{1} x \, dx \right) \, dy = \int_{y=0}^{2} y \left[ \frac{x^2}{2} \right]_{x=0}^{1} \, dy = \int_{y=0}^{2} y \frac{1}{2} \, dy$$

2. Evaluate remaining (single) integral:

$$\int_{y=0}^{2} y \frac{1}{2} \, dy = \left[ \frac{y^2}{4} \right]_{y=0}^{2} = 1 - 0 = 1$$

Yes, $g(x, y)$ is a valid joint PDF because it integrates to 1.
Marginal distributions

Suppose $X$ and $Y$ are continuous random variables with joint PDF:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1$$

The marginal density functions (marginal PDFs) are therefore:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) \, dy$$

$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b) \, dx$$

$$p_X(a) = \sum_y p_{X,Y}(a, y)$$
Back to darts!

Match $X$ and $Y$ to their respective marginal PDFs:
Back to darts!

Match $X$ and $Y$ to their respective marginal PDFs:
Extra slides

If you want more practice with double integrals, I’ve included two exercises at the end of this lecture.
Joint CDFs
An observation: Connecting CDF to PDF

For a continuous random variable $X$ with PDF $f$, the CDF (cumulative distribution function) is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

The density $f$ is therefore the derivative of the CDF, $F$:

$$f(a) = \frac{d}{da} F(a)$$

(Fundamental Theorem of Calculus)
Joint cumulative distribution function

For two random variables $X$ and $Y$, there can be a **joint cumulative distribution function** $F_{X,Y}$:

$$F_{X,Y}(a, b) = P(X \leq a, Y \leq b)$$

For discrete $X$ and $Y$:

$$F_{X,Y}(a, b) = \sum_{x \leq a} \sum_{y \leq b} p_{X,Y}(x, y)$$

For continuous $X$ and $Y$:

$$F_{X,Y}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y) dy \, dx$$

$$f_{X,Y}(a, b) = \frac{\partial^2}{\partial a \, \partial b} F_{X,Y}(a, b)$$
Single variable CDF, graphically

\[ f_X(x) \]

\[ \lim_{x \to -\infty} F_X(x) = 0 \]

\[ F_X(x) = P(X \leq x) \]

\[ \lim_{x \to +\infty} F_X(x) = 1 \]
Joint CDF, graphically

\[
\lim_{x,y \to -\infty} F_{X,Y}(x,y) = 0
\]

\[
\lim_{x,y \to +\infty} F_{X,Y}(x,y) = 1
\]

\[
f_{X,Y}(x, y)
\]

\[
F_{X,Y}(x, y) = P(X \leq x, Y \leq y)
\]
Independent Continuous RVs
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Proof of PDF:

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y)$$
$$= \frac{\partial}{\partial x} \frac{\partial}{\partial y} F_X(x)F_Y(y) = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$
$$= f_X(x)f_Y(y)$$
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

More generally, $X$ and $Y$ are independent if joint density factors separately:

$$f_{X,Y}(x,y) = g(x)h(y), \text{ where } -\infty < x, y < \infty$$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x, y) = 24xy$
   where $0 < x + y < 1$

$f_{X,Y}(x, y) = g(x)h(y)$, where $-\infty < x, y < \infty$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$
   Separable functions: $g(x) = 3e^{-3x}$
   $h(y) = 2e^{-2y}$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$
   Separable functions: $g(x) = 2x$
   $h(y) = 2y$

3. $f_{X,Y}(x, y) = 24xy$
   where $0 < x + y < 1$
   Cannot capture constraint on $x + y$
   into factorization!

If you can factor densities over all of the support, you have independence.
Bivariate Normal Distribution
Bivariate Normal Distribution

$X_1$ and $X_2$ follow a bivariate normal distribution if their joint PDF $f$ is

$$f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left( \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1 \sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)}$$

Can show that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

(Ross chapter 6, example 5d)

Often written as:

- Vector $X = (X_1, X_2)$
- Mean vector $\mu = (\mu_1, \mu_2)$, Covariance matrix: $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$

Recall correlation: $\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}$

We will focus on understanding the shape of a bivariate Normal RV.
These darts were actually thrown according to a bivariate normal distribution:

$$(X, Y) \sim \mathcal{N} (\mu, \Sigma)$$

$$\mu = (450, 600)$$

$$\Sigma = \begin{bmatrix} 900^2/4 & 0 \\ 0 & 900^2/25 \end{bmatrix}$$

Marginal PDFs:

$$X \sim \mathcal{N} \left( 450, \frac{900^2}{4} \right)$$

$$Y \sim \mathcal{N} \left( 600, \frac{900^2}{25} \right)$$
A diagonal covariance matrix

Let \( X = (X_1, X_2) \) follow a bivariate normal distribution \( X \sim \mathcal{N}(\mu, \Sigma) \), where

\[
\mu = (\mu_1, \mu_2), \quad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}
\]

Are \( X_1 \) and \( X_2 \) independent?

\[
f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left( \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)}
\]

\[
= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left( \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)}
\]

(Note covariance: \( \rho \sigma_1 \sigma_2 = 0 \))

\( \rho \sim \mathcal{U} \)

\( X_1 \) and \( X_2 \) are independent with marginal distributions

\( X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2) \)
16: Continuous Joint Distributions (I)

Lisa Yan
May 11, 2020

\[ \mathbb{E}[X] \]
Jointly continuous RVs

$X$ and $Y$ are jointly continuous if they have a joint PDF:

$$f_{X,Y}(x, y) \text{ such that } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \, dx = 1$$

Most things we’ve learned about discrete joint distributions translate:

Marginal distributions

$$p_X(a) = \sum_y p_{X,Y}(a, y) \quad f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy$$

Independent RVs

$$p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Expectation (e.g., LOTUS)

$$E[g(X, Y)] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y) \quad E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f_{X,Y}(x, y) dy \, dx$$

...etc.
Think

Slide 35 has a question to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Think by yourself: 2 min
Warmup exercise

$X$ and $Y$ have the following joint PDF:

$$f_{X,Y}(x, y) = 3e^{-3x}$$
where $0 < x < \infty, 1 < y < 2$

1. Are $X$ and $Y$ independent?

2. What is the marginal PDF of $X$? Of $Y$?

3. What is $E[X + Y]$?
Warmup exercise

$X$ and $Y$ have the following joint PDF:

$$f_{X,Y}(x, y) = 3e^{-3x}$$

where $0 < x < \infty$, $1 < y < 2$

1. Are $X$ and $Y$ independent?

2. What is the marginal PDF of $X$? Of $Y$?

3. What is $E[X + Y]$?

\[ f_{X,Y}(x, y) = \begin{cases} 3e^{-3x} & \text{for } 0 < x < \infty, 1 < y < 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ g(x) = 3Ce^{-3x}, \quad 0 < x < \infty \quad C \text{ is a constant} \]

\[ h(y) = \frac{1}{C}, \quad 1 < y < 2 \]

\[ f_{X,Y}(x, y) = \begin{cases} 3e^{-3x} & \text{for } 0 < x < \infty, 1 < y < 2 \\ 0 & \text{otherwise} \end{cases} \]
Warmup exercise

\(X\) and \(Y\) have the following joint PDF:

\[
f_{X,Y}(x, y) = 3e^{-3x}
\]

where \(0 < x < \infty, 1 < y < 2\)

1. Are \(X\) and \(Y\) independent?

2. What is the marginal PDF of \(X\)? Of \(Y\)?

3. What is \(E[X + Y]\)?

**Strategy 1:**

\[
E[g(X,Y)] = \int_0^{2\infty} g(x,y) f_{X,Y}(x,y) \, dx \, dy
\]

**Strategy 2:**

\[
E[X + Y] = E[X] + E[Y] = \frac{1}{3} + \frac{3}{2}
\]
Check out the question on the next slide (Slide 39). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Breakout rooms: 4 min. Introduce yourself!
The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define \( X = \) # minutes past 12pm that person 1 arrives. \( X \sim \text{Unif}(0, 30) \)

\( Y = \) # minutes past 12pm that person 2 arrives. \( Y \sim \text{Unif}(0, 30) \)

What is the probability that the first to arrive waits >10 mins for the other?

Compute: \( P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y) \) (by symmetry)

1. What is “symmetry” here?

2. How do we integrate to compute this probability?

\[
\int_{x, y} (\frac{1}{30})^2 \, dx \, dy + \int_{x, y} (\frac{1}{30})^2 \, dx \, dy
\]

\[
\frac{1}{120} \]

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The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define $X = \# \text{ minutes past 12pm that person 1 arrives.} \quad X \sim \text{Unif}(0, 30)$

$Y = \# \text{ minutes past 12pm that person 2 arrives.} \quad Y \sim \text{Unif}(0, 30)$

What is the probability that the first to arrive waits >10 mins for the other?

Compute: $P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y)$ \hfill (by symmetry)

\[
= 2 \cdot \int_{x+10<y} \int f_{X,Y}(x,y) dx \, dy = 2 \cdot \int_{x+10<y, \ 0 \leq x,y \leq 30} (1/30)^2 dx \, dy \quad \text{(independence)}
\]

\[
= \frac{2}{30^2} \int_{10}^{30} \int_0^{y-10} dx \, dy = \frac{2}{30^2} \int_{10}^{30} (y - 10) \, dy = \ldots = \frac{4}{9}
\]
Interlude for jokes/announcements
Announcements

Mid-quarter feedback form
link
Open until: this Friday

Problem Set 4
Due: Monday 5/18 10am
Covers: Up to and including today
Announcements: CS109 contest

Do something cool and creative with probability

Replaces one “passing” work requirement

Optional Proposal: Sat. 5/23 11:59pm
Due: Monday 6/8, 11:59pm

Interesting probability news

What That Election Probability Means

Even when you shift the probability far left or far right, the opposing candidate still gets some wins. That doesn’t mean a forecast was wrong. That’s just randomness and uncertainty at play. The probability estimates the percentage of times you get an outcome if you were to do something multiple times.

https://flowingdata.com/2016/07/28/what-that-election-probability-means/  
CS109 Current Events Spreadsheet
Bivariate Normal Distribution

The bivariate normal distribution of $X = (X_1, X_2)$:

$$X \sim \mathcal{N}(\mu, \Sigma)$$

- Mean vector $\mu = (\mu_1, \mu_2)$
- Covariance matrix: $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$

$$\text{Cov}(X_1, X_2) = \text{Cov}(X_2, X_1) = \rho \sigma_1 \sigma_2$$

- Marginal distributions: $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
- For bivariate normals in particular, $\text{Cov}(X_1, X_2) = 0$ implies $X_1, X_2$ independent.

We will focus on understanding the **shape** of a bivariate Normal RV.
Check out the question on the next slide (Slide 47). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Breakout rooms: 3 min. Introduce yourself!
(X, Y) Matching (all have $\mu = (0, 0)$)

1. PDF

2. PDF

3. PDF

4. PDF

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$
(X, Y) Matching (all have $\mu = (0, 0)$)

1. PDF

2. PDF

3. PDF

4. PDF

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

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Probabilities from joint CDFs

Recall for a single RV $X$ with CDF $F_X$:

$$P(a < X \leq b) = F_X(b) - F(a)$$

For two RVs $X$ and $Y$ with joint CDF $F_{X,Y}$:

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$

$$F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(a_2,b_1) + F_{X,Y}(a_1,b_1)$$

Note strict inequalities; these properties hold for both discrete and continuous RVs.
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =
F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(a_2,b_1) + F_{X,Y}(a_1,b_1)
\]
Proabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[ F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \\
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = 
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = 
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probability with Instagram!

(for next time)

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.
Gaussian blur

(for next time)

In a Gaussian blur, for every pixel:

• Weight each pixel by the probability that $X$ and $Y$ are both within the pixel bounds
• The weighting function is a Bivariate Gaussian (Normal) standard deviation parameter $\sigma$

Gaussian blurring with $\sigma = 3$:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2 \cdot 3^2}$$

Weight matrix:

Center pixel: (0, 0)
Pixel bounds:

$-0.5 < x \leq 0.5$
$-0.5 < y \leq 0.5$

What is the weight of the center pixel?

$$P(-0.5 < X \leq 0.5, -0.5 < Y \leq 0.5) =$$

$$= 0.206$$
Extra
1. Integral practice

Let $X$ and $Y$ be two continuous random variables with joint PDF:

$$f(x, y) = \begin{cases} 
4xy & 0 \leq x, y \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

What is $P(X \leq Y)$?

$$P(X \leq Y) = \int_{x \leq y, 0 \leq x, y \leq 1} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{y} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{y} 4xy \, dx \, dy$$

$$= \int_{y=0}^{1} 4y \left[ \frac{x^2}{2} \right]_{0}^{1} \, dy = \int_{y=0}^{1} 2y^3 \, dy = \left[ \frac{2}{4} y^4 \right]_{0}^{1} = \frac{1}{2}$$

$$P(X \leq Y) + P(X > Y) = 1 \Rightarrow P(X \leq Y) = 1/2$$
2. How do you integrate over a circle?

P(dart hits within $r = 10$ pixels of center)?

$$P(X^2 + Y^2 \leq 10^2) = \int \int f_{X,Y}(x, y) \, dy \, dx$$

Let’s try an example that doesn’t involve integrating a Normal RV 😊.
2. Imperfection on Disk

You have a disk surface, a circle of radius $R$. Suppose you have a single point imperfection uniformly distributed on the disk.

What are the marginal distributions of $X$ and $Y$? Are $X$ and $Y$ independent?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \leq R^2} dy = \frac{1}{\pi R^2} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2}$$

where $-R \leq x \leq R$.

$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$

No, $X$ and $Y$ are dependent. $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$.